Dynamics of an Ellipsoid on a Rough Surface for Two Cases of Friction

Maksim S. Salimov and Vasily E. Pogreev
National Research University “Moscow Power Engineering Institute”
Department of Robotics, Mechatronics, Dynamics and Strength of Machines, Moscow, Russia
Email: imaxsalimov@gmail.com, vasily.vep@gmail.com

***Abstract*—The movement of an ellipsoid with a displaced center of mass is an unusual phenomenon. When it moves on a horizontal plane relative to the vertical axis, the ellipsoid rapidly slows down its rotation. Then there are oscillations around other axes and then the rotation relative to the vertical axis is restored, but the new rotation occurs in a different direction. This phenomenon is similar to the movement of a semi-ellipsoid, the so-called “Celtic stone”. In this paper, we examine the dynamics of an axisymmetric ellipsoid on a rough surface. The contact between the body and the plane is determined by forces that take into account sliding and spinning in an associated form. The contact of the ellipsoid and the surface is considered as a small area (point) concerning the size of the body. This allows us to describe the contact with a combined two-dimensional friction model, and also to neglect the moment of dry friction forces relative to this point. The center of mass of the ellipsoid lies on the axis of its dynamic symmetry and is displaced by some distance from the geometric center of the ellipsoid. This nonuniform distribution of the ellipsoid's mass allows these unusual phenomena to appear because the ellipsoid tends to rotate in the direction where its mass is excessive. The integral expression for the friction force is replaced by Pade approximations and then the corresponding expansion coefficients are determined. The dependence of the found friction force on the sliding and rotation velocities is plotted depending on both arguments.**

***Index Terms*—ellipsoid, dynamics, Solid, Dry friction, Padé approximations, sliding, spin, rotation, horizontal surface**

I. INTRODUCTION

Some past researches of the movement of the nonhomogeneous ellipsoid [1] are based on assumptions that simplify and at the same time limit the possibility of explaining the movement by physical considerations. For example, an assumption such as the absence of velocity of the point of contact with the horizontal plane limits further explanation.

This non-holonomic formulation of the problem [1] is far from reality. Thus, it is necessary to set a problem about the movement of the nonhomogeneous ellipsoid, which would correspond to real physical considerations. The motion of the ellipsoid in this paper is based on a special interaction of the ellipsoid with the horizontal plane. In this interaction, the contact is set by forces that account for sliding and spin in a related form. The use of this interaction can serve for numerous studies of the movements of different bodies.

Numerous investigations of this phenomenon have been conducted in the following articles [2], [3], [4], [5], [6]. For example, in [2], the authors pay attention to the rolling resistance model and analyze it in detail. The paper [5] focuses on modeling and experimental verification of rotation and rolling of a semi-ellipsoid. The paper [6] focuses on the investigation of the stationary motion of an ellipsoid, and the authors also give a geometric interpretation of the results.

II. FORMULATION OF THE PROBLEM

A solid is an ellipsoid with a displaced center of mass that moves along a horizontal plane Oxy. The ellipsoid is supported on the bearing plane by a convex surface, at each point of which the normal n is uniquely defined.

$Oxyz$ is a fixed coordinate system with an origin on the bearing surface. $Px_1y_1z_1$ is a moving coordinate system with the origin in the center of mass of the ellipsoid and the axes that are directed along the main axes of inertia of the body. Point G is the point of contact between the solid and the surface. a, c are the equatorial and axial radii of the ellipsoid, respectively, with $a > c$. The center of mass of the ellipsoid P lies on the axis of its dynamic symmetry and is displaced along this axis from the geometric center of the ellipsoid O_1 by a distance δ. If $\delta > 0$, then the contact point G lies on the negative part of the z_1 axis. $\bar{\gamma}$ is the vertical unit vector Oz, g is the acceleration of gravity, m is the mass of the body, N is the reaction of the bearing plane ($N = mg$).

The radius vector $\bar{r}$ is the distance from the center of mass P to the point of contact G. The radius vector $\bar{r}$ is determined in the coordinate system $Px_1y_1z_1$ by the following components:

$$r_1 = -\frac{a^2\gamma_1}{s}, \qquad r_2 = -\frac{a^2\gamma_2}{s}, \qquad r_3 = \delta - \frac{c^2\gamma_3}{s}$$
$$s = \sqrt{(c^2 - a^2)\gamma_3^3 + a^2}$$

Manuscript received September 18, 2020; revised December 1, 2020.
The investigation was carried out within the framework of the project “Quantitative expression of Mohr's theory of strength” with the support of a grant from NRU "MPEI" for implementation of scientific research programs "Energy", "Electronics, Radio Engineering and IT", and “Industry 4.0, Technologies for Industry and Robotics in 2020-2022”

doi: 10.18178/ijmerr.10.8.452-457

where $\gamma_1, \gamma_2, \gamma_3$ are unit vectors of the coordinate system $Px_1y_1z_1$.

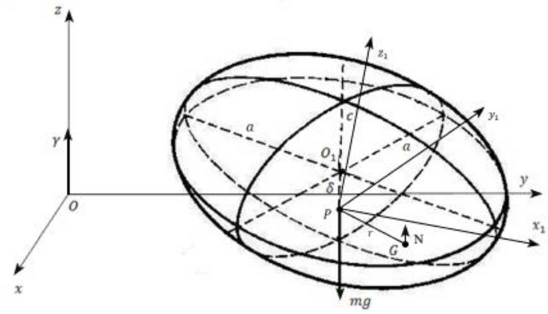


Figure 1. The movement of an ellipsoid on the surface.

The main axes of inertia $Px_1y_1z_1$ are associated with an ellipsoid with a beginning in the geometric center of the body. The ellipsoid is defined by the equation of the surface in the axes $Px_1y_1z_1$:

$$\Phi(x) = \frac{{x_1}^2 + {y_1}^2}{a^2} + \frac{(z_1 - \delta)^2}{c^2} - 1 = 0, \qquad (1)$$

where φ is the angle between the largest axis of the ellipsoid and the main axis of inertia x_1.

The equations of motion of the body referred to the moving coordinate system $Px_1y_1z_1$:

$$m\frac{d\bar{v}}{dt} + \bar{\omega} \times m\bar{v} = (\bar{N} - mg)\bar{\gamma} - \bar{F}, \qquad (2)$$

$$\Theta\frac{d\bar{\omega}}{dt} + \bar{\omega} \times (\Theta\bar{\omega}) = \bar{r} \times (\bar{N}\bar{\gamma} - \bar{F}), \qquad (3)$$

$$\frac{d\bar{\gamma}}{dt} + \bar{\omega} \times \bar{\gamma} = 0, \qquad (4)$$

One needs to add the holonomic coupling equation, which is the condition for the contact of the ellipsoid with the reference plane:

$$(\bar{v} + \bar{\omega} \times \bar{r}) \cdot \bar{\gamma} = 0, \qquad (5)$$

where v is the velocity of the center of mass of the ellipsoid P, $\bar{\omega}$ is the angular velocity of the ellipsoid, $\Theta = \mathrm{diag}(A_1, A_1, A_3)$ is the central inertia tensor, $\bar{F}$ is the friction force applied to the body at the contact point G. $v = \bar{v} + \bar{\omega} \times \bar{r}$ is the slip velocity of the ellipsoid.

The obtained equations of motion express the theorem on changes in the momentum of a body (2), the theorem on the change in the kinetic moment of motion of the body (3), the condition for the constancy of the unit vector $\bar{\gamma}$ in the fixed reference system $Oxyz$ (4) and the condition that the ellipsoid contacts the bearing plane (5).

The system of equations of motion (2) - (5) is closed relative to variables $\bar{v}$, $\bar{\omega}$, $\bar{\gamma}$ and $\bar{F}$ and admits an integral of the total mechanical energy and a geometric integral.

$$\begin{cases} H = \frac{m\bar{v}^2}{2} + \frac{(\Theta\bar{\omega}, \bar{\omega})}{2} - mg(\bar{r}, \bar{\gamma}) = h = const \\ \gamma^2 = 1 \end{cases}, \qquad (6)$$

To solve the system of equations of motion (2) - (5), it is necessary to obtain the dependence between the vector $\bar{r}$ and the unit vector $\bar{\gamma}$. The equation of the body surface will help to determine this relationship. If it is $f(\bar{r}) = 0$, where $grad\, f(\bar{r})$ is directed outward relative to the contact area of the body. Then

$$\bar{\gamma} = -\frac{grad\, f(\bar{r})}{|grad\, f(\bar{r})|}, \qquad (7)$$

Let's assume that the friction force is given as $\bar{F} = F(v, \omega, \gamma, N)$. Further, we consider two cases: a combined model of dry friction and a model without slipping.

III. Combined Model of Dry Friction During Rolling Motion

The combined model of dry friction begins with investigations of the rolling of a solid, where the contact of the body and the plane is described by the forces of dry friction [7]. A two-dimensional friction model can be applied to the nonhomogeneous ellipsoid. Therefore, one can consider the contact area as a point and neglect the moment of dry friction forces relative to this point.

According to the Hertz theory of contact stresses [8], solids take a spherical shape over the contact area. At different points of this contact, slippage is different. Therefore, one can assume that the contact surface is locally spherical. The normal stress there depends only on the distance ρ to the center of the circle. In this case, the contact occurs over a small circular region of radius ε, which depends on the elastic modulus of the material and the load N. Relative slipping occurs at the speed v. The relative slip velocity v_G at the point G is determined by the polar coordinates ρ and θ:

$$v_G = (v - \omega\rho sin\theta, \omega\rho sin\theta)$$

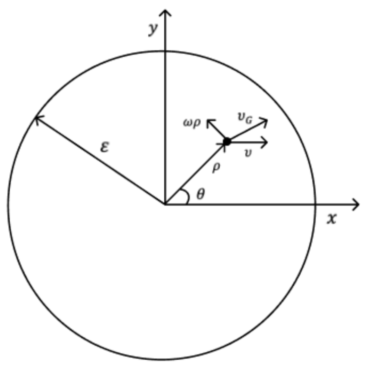


Figure 2. Spherical area of the contact spot.

The differential of the friction force is directed against the relative velocity v_G. In accordance with the law of Coulomb, it is equal to

$$dF = -f\sigma(\rho)\frac{v_G}{|v_G|}dS, \qquad (8)$$

Differential moment of this force:

$$dM = -|\rho \times dF| = -\rho_x dF_y + \rho_y dF_x = \frac{f\sigma(\rho)\rho(\rho\omega - vsin\theta)}{|v_G|}dS$$

The relative velocity module v_G is defined as

$$|v_G| = \sqrt{(\rho\omega)^2 + (v - \omega\rho sin\theta)^2} = \sqrt{\rho^2\omega^2 + v^2 - 2\omega v\rho sin\theta}$$

Substituting the speed module and solving the differential:

$$F = -f\int_0^1\int_0^{2\pi} \frac{(v - \omega\rho sin\theta, \omega\rho sin\theta)\sigma(\rho)\rho}{\sqrt{\rho^2\omega^2 + v^2 - 2\omega v\rho sin\theta}} d\rho d\theta$$

$$M = -f\int_0^1\int_0^{2\pi} \frac{(\rho\omega - v sin\theta)\sigma(\rho)\rho^2}{\sqrt{\rho^2\omega^2 + v^2 - 2\omega v\rho sin\theta}} d\rho d\theta$$

where the first integral is determined at a distance from the center of the circular contact area to $\rho = 1$.

The component relative to the x_1 axis in the equation for the force is zero along the y_1 axis, due to symmetry.

The obtained equations with simplifications $u = \omega\varepsilon$ and $\tau = \rho/\varepsilon$:

$$F(u,v) = f\varepsilon^2\int_0^1 \tau\,\sigma(\tau)\int_0^{2\pi} \frac{v - u\tau\,sin\theta}{\sqrt{u^2\tau^2 + v^2 - 2uv\tau sin\theta}} d\theta d\tau, \quad (9)$$

$$M(u,v) = f\varepsilon^3\int_0^1 \tau\,\sigma(\tau)\int_0^{2\pi} \frac{u\tau^2 - vr\tau\,sin\theta}{\sqrt{u^2\tau^2 + v^2 - 2uv\tau sin\theta}} d\theta d\tau, \quad (10)$$

Therefore, as $\varepsilon \to 0$, the friction moment $M(u,v) \to 0$, and the friction force does not depend on this variable.

The distribution of normal stresses at the contact point is expressed by the Hertz formula [9]:

$$\bar{\sigma} = \frac{3N}{2\pi\varepsilon^2}\sqrt{1 - \frac{\rho^2}{\varepsilon^2}} = \frac{3N}{2\pi\varepsilon^2}\sqrt{1 - \tau^2}, \quad (11)$$

Substituting the formula (9) into (11):

$$F(u,v) = \frac{3Nf}{2\pi}\int_0^1\int_0^{2\pi} \tau\sqrt{1 - \tau^2}\frac{v - u\tau\,sin\theta}{\sqrt{u^2\tau^2 + v^2 - 2uv\tau sin\theta}} d\theta d\tau, \quad (12)$$

The graph of the friction force is shown in Fig. 3.

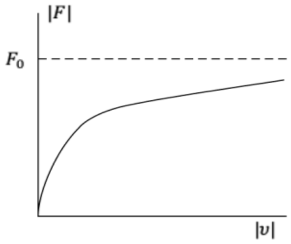


Figure 3. The dependence of the friction force on the sliding speed.

The ratio (12) is a linear surface. Functions (9) and (10) are homogeneous zero-order functions that can be converted to similar functions. Thus, these functions can be defined by the limit of relations of two multilinear functions of the same order. If the multilinear functions tend to infinity [10].

The integrals from equation (12), after their differentiation:

$$\frac{\partial F}{\partial v}|_{v=0} = \frac{3fN}{2\pi u}\int_0^1\int_0^{2\pi} \sqrt{1 - \tau^2}(1 - sin^2\theta)\,d\theta d\tau = \frac{3fN}{2\pi u}\frac{\pi^2}{4} = \frac{3\pi fN}{8u} \quad (13)$$

$$\frac{\partial F}{\partial v}|_{u=0} = 0, \quad (14)$$

$$\frac{\partial F}{\partial u}|_{v=0} = \frac{3fN}{2\pi u}\int_0^{2\pi} \tau\sqrt{1 - \tau^2}sin\theta\,d\theta = 0, \quad (15)$$

$$\frac{\partial F}{\partial u}|_{u=0} = \frac{3fN}{2\pi v}\int_0^1\int_0^{2\pi} 2\tau\sqrt{1 - \tau^2}sin\theta\,d\theta d\tau = 0, \quad (16)$$

$$F|_{v\to\infty} = \frac{3fN}{2\pi u}\int_0^1 \tau\sqrt{1 - \tau^2}\,d\tau = \frac{fN}{2\pi}, \quad (17)$$

$$F|_{u\to\infty} = \frac{3fN}{2\pi v}\int_0^1\int_0^{2\pi} \tau\sqrt{1 - \tau^2}(-sin\theta)d\theta\,d\tau = 0, \quad (18)$$

The fractional-linear Padé approximation for the friction force equation (12) has the form:

$$F = \frac{a_1v + b_1u}{c_1v + d_1u} \quad (19)$$

In order to determine the Padé coefficients, it is necessary to study the properties of these expressions at the boundary points. One can differentiate (19) under different conditions and equate the obtained integral expressions (13) - (18):

$$\frac{\partial F}{\partial v}|_{v=0} = \frac{a_1d_1u - b_1uc_1}{d_1^2u^2} = \frac{3\pi fN}{8u},$$

$$\frac{\partial F}{\partial v}|_{u=0} = \frac{a_1c_1 - a_1c_1}{vc_1^2} = 0,$$

$$\frac{\partial F}{\partial u}|_{v=0} = \frac{b_1ud_1 - b_1ud_1}{d_1^2u^2} = 0,$$

$$\frac{\partial F}{\partial u}|_{u=0} = \frac{b_1c_1 - a_1d_1}{vc_1^2} = 0,$$

$$F|_{v\to\infty} = \frac{a_1}{c_1} = \frac{fN}{2\pi}, \qquad F|_{u\to\infty} = \frac{b_1}{d_1} = 0,$$

There are four unknowns a_1, b_1, c_1c_1 and d_1 and three equations.

$$\frac{a_1d_1u - b_1uc_1}{d_1^2u^2} = \frac{3\pi fN}{8u}, \quad \frac{a_1}{c_1} = \frac{fN}{2\pi}, \quad \frac{b_1}{d_1} = 0 \quad (20)$$

Simplified expression (20) of three unknowns is

$$F = \frac{a_1v + b_1u}{c_1v + d_1u} = \left(\frac{a_1}{c_1}\right)\frac{v + (b_1/a_1)\,u}{v + (d_1/c_1)\,u} = F_0\frac{v + \tilde{b}_1u}{v + \tilde{d}_1u} \quad (21)$$

where $F_0 = a_1/c_1$, $\tilde{b}_1 = b_1/a_1$и$\tilde{d}_1 = d_1/c_1$. Substituting the obtained transformations into the system (20), there are the final Padé coefficients:

$$\begin{cases} \tilde{d}_1 = \dfrac{4}{3\pi^2} \\ F_0 = \dfrac{fN}{2\pi}, \\ \tilde{b}_1 = 0 \end{cases}$$

Thus, there is a formula for the friction force in the form:

$$F = \frac{fNv}{2\pi\left(v + \frac{4}{3\pi^2}u\right)}, \qquad (21.1)$$

Graph of the resulting friction force as a function of both arguments of the sliding speed and the speed of rotation:

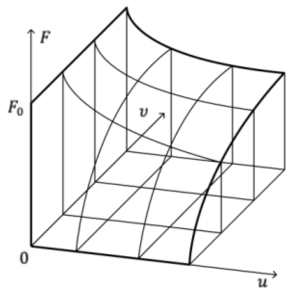


Figure 4. Dependence of the friction force on the sliding speed and the speed of rotation.

Comparing Fig. 3 and 4, we can conclude about the same behavior of the friction force.

Formula (21.1) in vector form is

$$\bar{F} = \frac{fN(\bar{v} + \bar{\omega} \times \bar{r})}{2\pi|\bar{v} + \bar{\omega} \times \bar{r}| + \frac{8}{3\pi}|\varepsilon\bar{\omega} \cdot \bar{\gamma}|}, \qquad (22)$$

A. Analysis of the Motion of an Ellipsoid in a Combined Model

Ellipsoid motion analysis is performed using Wolfram Mathematica. The main parameters of the ellipsoid: $m = 0.14$ kg, $A_1 = 0.002$ kg m^2, $A_3 = 0.0006$ kg m^2, $a = 0.1$ m, $c = 0.03$ m, $g = 10$ m/s^2, $f = 0.1$, $\varepsilon = 0.004$ m, $\delta = 0{,}002$ m. Initial parameters: $v_1(0) = 0$, $v_2(0) = 0$, $\gamma_1(0) = 0$, $\gamma_2(0) = 0$, $\gamma_3(0) = \pm 1$.

The figure of the angular velocity $\omega_3(t)$ in the case when the angle φ and $\omega_3(0)$ has opposite signs, for $\varphi = -0{,}2$, $\omega_1(0) = 0{,}1$ m/s $\omega_2(0) = 0{,}3$ m/s, $\omega_3(0) = 1$ m/s, $T = 2000$ s.

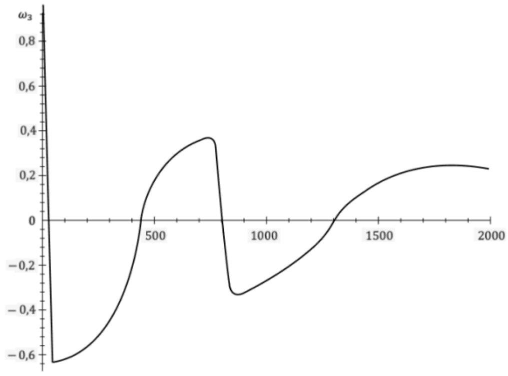


Figure 5. Angle φ and $\omega_3(0)$ with opposite signs.

Fig. 5 shows a sharp decrease in angular velocity and subsequent change of sign. This indicates a change in direction of rotation. This figure is in good agreement with the experimental results [7].

The figure of the angular velocity $\omega_3(t)$ in the case when the angle φ and $\omega_3(0)$ have identical signs, for $\varphi = 0.2$, $\omega_1(0) = 0.1$ m/s $\omega_2(0) = 0.3$ m/s, $\omega_3(0) = 1$ m/s, $T = 2000$ s.

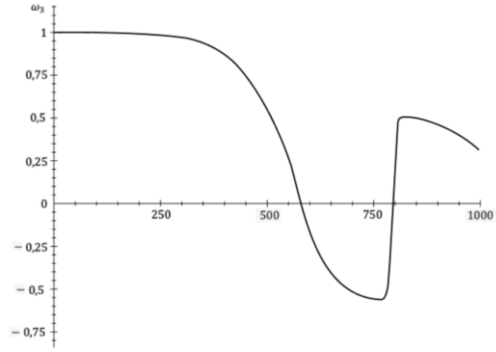


Figure 6. Angle φ and $\omega_3(0)$ with identical signs.

The character of the change in $\omega_3(0)$ differs utterly from the first case, the rotational motion is preserved in one direction for a long time. The obtained figures of the ellipsoid motion model provide adequate descriptions of the motion [6].

IV. Movement without Slippage

The friction force satisfies the conditions $\bar{F} = 0$ and $\bar{v} = \bar{\omega} \times \bar{r}$. There is no rest friction force unless there is any small spin. The equations of motion (2) and (3) take the form:

$$m\left(\frac{d\bar{r}}{dt} \times \bar{\omega}\right) + m\left(\bar{r} \times \frac{d\bar{\omega}}{dt}\right) + m(\bar{\omega} \times (\bar{r} \times \bar{\omega})) = (N - mg)\bar{\gamma}, \qquad (23)$$

$$\Theta\frac{d\bar{\omega}}{dt} + \bar{\omega} \times (\Theta\bar{\omega}) = N\bar{r} \times \bar{\gamma}, \qquad (24)$$

Based on (24) there is a condition

$$\omega_3 = const, \qquad (25)$$

Equations (25) and (4) admit the existence of an integral

$$A_1\omega_1\gamma_1 + A_2\omega_2\gamma_2 + A_3\omega_3\gamma_3 = const, \qquad (26)$$

The terms of equations (23) and (24) in the projections on the inertia axis are

$$\bar{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}, \quad \Theta = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}, \quad \bar{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}, \quad \bar{\gamma} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}$$

The transformation of formulas (23), (24) and the substitution of the values of the components of the radius vector $\bar{r}$ determine the equations of motion in general form

$$\left(\frac{d\omega_1}{dt}\gamma_1 + \frac{d\omega_2}{dt}\gamma_2\right)\left(\delta - \frac{c^2\gamma_3}{s}\right) + \left(\frac{c^2 - a^2}{s^3}\right) \cdot$$

$$\cdot\left(\left(c^2\gamma_3^3\omega_3 - a^2(1 - \gamma_3^2 - \delta)(\omega_2\gamma_2 + \omega_1\gamma_1)\right) - \delta\omega_3\right) = 0, \qquad (27)$$

$$A_1\left(\frac{d\omega_1}{dt}\gamma_1 + \frac{d\omega_2}{dt}\gamma_2\right) + (\omega_2\gamma_1 - \omega_1\gamma_2(A_3 - A_1))\omega_3 = 0, \qquad (28)$$

Based on equation (28), there is a condition $\omega_2\gamma_1 - \omega_1\gamma_2 = 0$, from which there is another condition $\gamma_3 = const$.

The system of equations of motion (27), (28), and (4) with $\gamma_3^2 = 1 + \delta$ has a solution. This solution corresponds to uniform rotations of the ellipsoid relative to the axis $O_1 z_1$ for different locations of the center of mass $\gamma_3 = \pm\sqrt{1+\delta}$:

$$\gamma_1 = \gamma_2 = 0, \quad \omega_1 = \omega_2 = 0, \quad \omega_3 = \omega = const, \quad N = mg, \tag{29}$$

B. Analysis of the Motion of an Ellipsoid without Slipping

For numerical integration, it is necessary to specify additional parameters of the ellipsoid in the case of viscous friction. Viscous friction is characterized by a viscosity coefficient ν =3 Ns/m and is set by the system:

$$F = \begin{cases} -\nu(\bar{v} + \bar{\omega} \times \bar{r}), & N \neq 0 \\ 0, & N = 0 \end{cases}, \tag{30}$$

For numerical integration, the bearing surface is absolutely rigid. During the separation of the ellipsoid from the bearing plane, the reaction force is N = 0, separation occurs when the body overturns. At this point in time, the integration of the system of equations (2) - (5) occurs under condition (30), when N = 0.

The center of mass of the ellipsoid is located lower by δ = 0.0025 m from the geometric center of the ellipsoid. The angular velocity ω is directed vertically upward and varies uniformly in the range from 10 to 50 s^{-1}. The unit vertical vector is $\gamma_3(0) = 0.9$.

The figures show the behavior of some quantities as a function of time t at the highest initial angular velocity of the body;

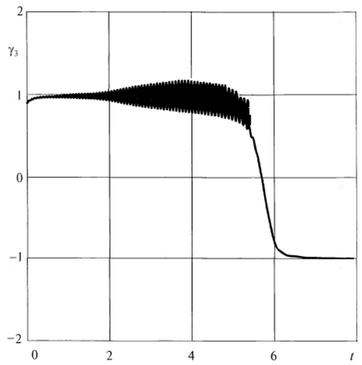


Figure 7. Change in the angle between the axes of proper and precessional rotation γ_3 (nutation angle) relative to time t.

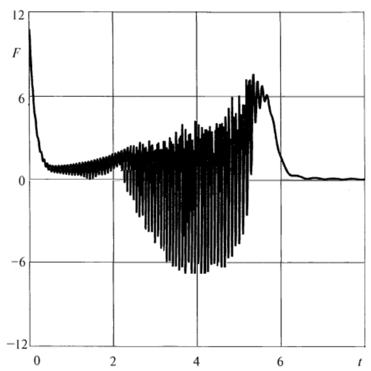


Figure 8. Change in friction force F relative to time t.

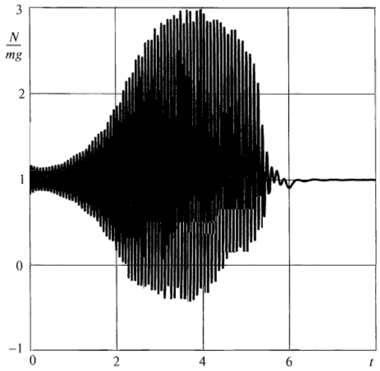


Figure 9. Change in the ratio of the normal component of the reaction of the support to body weight relative to time t.

Fig. 7 shows the ellipsoid overturn at 5-6 seconds. At this moment, the point G of the initial contact with the plane is above the geometric center of the ellipsoid O_1.

Fig. 8 demonstrates the change in the magnitude and direction of the friction force, which corresponds to numerous experiments [11], [12], [13]. About 6 seconds during the separation of the ellipsoid from the bearing plane, the viscous friction decreases to zero.

Fig. 9 shows strong changes in the support reaction after 2 seconds. This indicates rotation when the contact area is infinitely small. About 6 seconds, the ellipsoid overturns and separates from the reference plane, and the reaction force is reduced to zero.

V. CONCLUSIONS

The combined dry friction model is considered and applied when the contact of the nonhomogeneous ellipsoid and a flat surface is considered as a small area (point) compared to the size of the body. The resulting integral model of dry friction is consistent with the dynamics of the movement under consideration and describes the interaction of the body and the surface during sliding and spin. The fractional-linear Padé approximation was applied to resolve the integral. After finding the decomposition coefficients, the friction force was determined. The friction force is in qualitative agreement with the integral expression, based on figures with dependencies. The changes in the angular velocity relative to the vertical axis are graphically illustrated for different values of the angle between the largest axis of the ellipsoid and the main axis of inertia x_1. Based on these figures, one can conclude that the model of the motion of the ellipsoid has a physical character close to real.

The motion of the nonhomogeneous ellipsoid without slipping is investigated. The equations of motion are determined under the appropriate conditions of friction. The graphs of different dependencies demonstrate the dynamics of the ellipsoid, which is consistent with numerous experiments [11], [12], [13].

REFERENCES

[1] A. P. Markeev, "Dynamics of a body in contact with a solid surface," *Science, Physics and Mathematics*, pp. 336-341, 1992.

[2] J. Awrejcewicz, and G. Kudra, "Rolling resistance modeling in the Celtic stone dynamics," *Multibody System Dynamics*, vol. 45, no. 2, pp. 155–167, 2019.

[3] A. S. Gonchenko, S. V. Gonchenko, A. O. Kazakov, and E. A. Samylina, "Chaotic dynamics and multistability in the nonholonomic model of a celtic stone," *Radiophysics and Quantum Electronics*, vol. 61, no. 10, pp. 773–786, 2019.

[4] A. V. Karapetyan, M. A. Municyna, "Dynamics of an inhomogeneous ellipsoid on a horizontal plane," *Applied Mathematics and Mechanics*, vol. 3, pp. 328-333, 2014.

[5] G. Kudra and J. Awrejcewicz, "Numerical and experimental investigation of the celtic stone," in *Proc. 8th European Nonlinear Dynamics Conference*, 2014, pp. 2-6.

[6] V. F. Zhuravlev and D. M. Klimov, "Global celtic stone movement," *Solid State Mechanics*, vol. 3, pp. 8-16, 2008.

[7] V. Andronov and V. Zhuravlev, (n.d.). *Dry Friction in Problems of Mechanics*, Moscow-Izhevsk, 2009, pp. 189.

[8] P. Purushothaman and P. Thankachan, "Hertz contact stress analysis and validation using finite element analysis," *International Journal for Research in Applied Science Engineering Technology*, vol. 2, 2014, pp. 531–538.

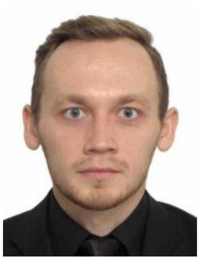
[9] A. Kireenkov, "Related model of rolling and sliding friction," *Reports of the Academy of Sciences*, vol. 6, pp. 750-755, 2011.

[10] I. Hassan, I. Youssef, T. Rageh, "New approaches for Taylor and Padé approximations," *International Journal of Advances in Applied Mathematics and Mechanics*, vol. 2, pp. 78-86, 2015.

[11] A. V. Borisov, A. O. Kazakov, and S. P. Kuznetsov, "Nonlinear dynamics of the rattleback: A nonholonomic model," *Physics-Progress*, vol. 57, no. 5, pp. 453–460, 2014.

[12] M. Hanias, S. G. Stavrinides, and S. Banerjee, "Analysis of rattleback chaotic oscillations," *The Scientific World Journal*, 2014.

[13] K. Yoichiro and N. Hiizu, "Rattleback dynamics and its reversal time of rotation," *Physical Review*, 2017.



Maksim Salimov is a postgraduate student of the Department of Robotics, Mechatronics, Dynamics and Strength of Machines at the National research university "MPEI". His main research interests are in the field of dynamics of solids, as well as the study of dry friction.

Vasily Pogreev is a master student of the Department of Robotics, Mechatronics, Dynamics and Strength of Machines at the National research university "MPEI". His main research interests are in the field of MEMS gyroscopes.