Mixed Finite Element Formulation for the Free Vibration Analysis of Viscoelastic Plates with Uniformly Varying Cross Sections

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***Abstract*—This paper is concerned with the free vibration analysis of viscoelastic thin plates of uniformly varying cross sections constituted with a four-parameter model under different time-dependent loadings. A mixed finite element formulation in the Laplace–Carson domain is used to construct a functional for the analysis of the viscoelastic plate of uniformly varying cross sections. The four-parameter model is taken into consideration in viscoelastic modeling. The Durbin inverse Laplace transform technique is utilized to obtain the viscoelastic solution in time domain. The developed solution technique is applied to several dynamic example problems to analyze the free vibration behavior of viscoelastic plates of uniformly varying cross sections.**

***Index Terms*—Four-parameter mechanical model, free vibration analysis, Kirchhoff plate theory, mixed finite element.**

I. Introduction

Although the static and dynamic responses of elastic plate, beam, and shell structures are widely studied topics, there are few studies that exist in the literature pertaining to the analysis of the viscoelastic structural elements, especially with complex geometries, loading conditions, and constitutive relations. For the problems that have complex geometries, loading conditions, and material properties, closed-form solutions are often not possible. Therefore, numerical solution methods should be employed. The application of the finite element method to the solution of viscoelastic problems has been presented by a number of authors [1, 2, 3].

Due to the fact that viscoelastic materials reflect the real material behavior, comprehensive studies on the behavior of viscoelastic materials have become very important, and an accurate description of the mathematical model to analyze structural components with viscoelastic materials is going to be necessary. The basic concepts of mechanical behavior of structural components with viscoelastic materials are explained [4, 5].

This paper is devoted to the analysis of the free vibration behavior of viscoelastic thin plates of uniformly varying cross sections under different time-dependent loadings. For the analysis, a viscoelastic thin plate is modeled by a four-parameter solid model. The Maxwell and Kelvin models are the simplest viscoelastic models. More realistic material responses can be modeled using more elements. In contrast to the other well-known simpler Maxwell and Kelvin–Voigt models, the four-parameter solid model is slightly more complex, involving Maxwell and Kelvin–Voigt elements in parallel. In order to obtain the constitutive relation of the viscoelastic plate material, new operators in the hereditary integral form are defined. In order to remove time derivatives from the governing equations, the Laplace–Carson transform is employed. A functional in the Laplace–Carson space is presented using the mixed finite element method based on the Gâteaux differential.

The studies [6, 7, 8, 9] are among the first to propose the mixed finite element solution model to analyze viscoelastic structural components. In these studies, mixed finite element formulations of viscoelastic Timoshenko, parabolic, and circular beams were derived on the basis of the Gâteaux differential method. Some significant advantages of this most powerful variational tool, when compared to the other famous ones such as Hellinger–Reissner and Hu–Washizu, were explained in detail in [10, 11, 12].

To the best of the authors' knowledge, this will be the first study that presents a functional for the free vibration analysis of viscoelastic Kirchhoff plates of uniformly varying cross sections constituted with a four-parameter model. The solutions obtained in the Laplace–Carson domain are converted to a real-time domain by the inverse Laplace transform via Durbin's algorithm [13]. The performance of the developed solution method is presented by several dynamic example problems.

II. Method

A. Governing Equations

For three-dimensional bodies, the constitutive equations of viscoelastic materials have two different operators for dilatation and distortion. Assuming that the dilatation and distortion parameters are equal corresponds to the assumption that Poisson's ratio is constant. Using this assumption, the stress couples of the Kirchhoff plates

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(two bending moments, M_x and M_y, and one twisting moment, M_{xy}) can be derived in terms of the plate's middle surface displacement (w) as follows:

$$M_x + D^* \left[\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right] = 0 ,$$
$$M_y + D^* \left[\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right] = 0 , \qquad (1)$$
$$M_{xy} + \left(1 - \nu\right) D^* \frac{\partial^2 w}{\partial x \partial y} = 0 ,$$

where ν is Poisson's ratio and D^* is the operator form of the flexural rigidity of the plate. In order to define the operator form of the flexural rigidity of the plate, a new operator in the hereditary integral form is defined as follows:

$$E^* f = E_{(0)} f_{(t)} + \int_0^t \frac{dE_{(t-\tau)}}{d\left(t-\tau\right)} f_{(\tau)} d\tau . \qquad (2)$$

Additional details about these operators can be obtained from [4] in a simple form and can be obtained from [11] for the plates. The governing equation of the viscoelastic plates is derived considering the equilibrium equations, kinematic relations, and constitutive relations as follows:

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = 0 . \qquad (3)$$

where q indicates normal load distribution. The positive sign convention for the stress couples is illustrated in Fig. 1.

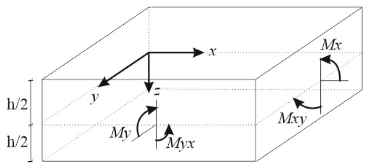

Figure 1. Stress couples.

In order to remove the time derivatives from the governing equations and boundary conditions, the method of Laplace–Carson transform will be employed. The Laplace–Carson transform of a real function is defined as an s-multiplied Laplace transform. Taking the Laplace–Carson transform of Eqs. (1) and (3), we will obtain the field equations in Laplace–Carson space:

$$-\bar{M}_x - \bar{D}^* \left(\frac{\partial^2 \bar{w}}{\partial x^2} + \nu \frac{\partial^2 \bar{w}}{\partial y^2} \right) = 0 ,$$
$$-\bar{M}_y - \bar{D}^* \left(\frac{\partial^2 \bar{w}}{\partial y^2} + \nu \frac{\partial^2 \bar{w}}{\partial x^2} \right) = 0 , \qquad (4)$$
$$-\bar{M}_{xy} - \left(1-\nu\right)\bar{D}^* \frac{\partial^2 \bar{w}}{\partial x \partial y} = 0 ,$$
$$-\frac{\partial^2 \bar{M}_x}{\partial x^2} - 2\frac{\partial^2 \bar{M}_{xy}}{\partial x \partial y} - \frac{\partial^2 \bar{M}_y}{\partial y^2} = \bar{q} ,$$

Field equations given in Eq. (4) can be written in an operator form as follows:

$$\bar{\mathbf{Q}} = \bar{\mathbf{L}}\,\bar{\mathbf{y}} - \bar{\mathbf{f}} . \qquad (5)$$

where $\bar{\mathbf{L}}$ represents the coefficient matrix, $\bar{\mathbf{y}}$ represents the unknown vector, and $\bar{\mathbf{f}}$ represents the load vector. An operator $\bar{\mathbf{Q}}$ in Laplace–Carson space will be a potential operator if the equality given by [14]

$$< d\bar{\mathbf{Q}}(\bar{\mathbf{y}}, \bar{\mathbf{y}}'), \bar{\mathbf{y}}^* > = < d\bar{\mathbf{Q}}(\bar{\mathbf{y}}, \bar{\mathbf{y}}^*), \bar{\mathbf{y}}' > . \qquad (6)$$

is satisfied. If the operator is potential, then the functional corresponding to the field equations will be obtained as follows:

$$\mathrm{I}(\bar{\mathbf{y}}) = \int_0^1 \left[\bar{\mathbf{Q}}(s\bar{\mathbf{y}}), \bar{\mathbf{y}} \right] ds . \qquad (7)$$

where s is a scalar quantity. Using Eq. (7), the functional corresponding to the operator in the Laplace–Carson domain becomes

$$\begin{aligned} \mathrm{I}(\bar{\mathbf{y}}) &= \left[\bar{w}_{,x}, \bar{M}_{x_{,x}} \right] + \left[\bar{w}_{,y}, \bar{M}_{y_{,y}} \right] + \left[\bar{w}_{,x}, \bar{M}_{xy_{,y}} \right] \\ &+ \left[\bar{w}_{,y}, \bar{M}_{xy_{,x}} \right] - \left[\bar{q}, \bar{w} \right] - \frac{1}{2\bar{D}\,(1-\nu^2)} \left\{ \left[\bar{M}_x, \bar{M}_x \right] + \left[\bar{M}_y, \bar{M}_y \right] \right\} \\ &+ \frac{\nu}{\bar{D}\,(1-\nu^2)} \left[\bar{M}_x, \bar{M}_y \right] - \frac{1}{\bar{D}\,(1-\nu)} \left[\bar{M}_{xy}, \bar{M}_{xy} \right] - \left[\hat{\bar{T}}, \bar{w} \right]_\sigma \\ &- \left[\left(\bar{M} - \hat{\bar{M}} \right), \bar{w}' \right]_\sigma - \left[\hat{\bar{w}}', \bar{M} \right]_\varepsilon - \left[\left(\bar{w} - \hat{\bar{w}} \right), \bar{T} \right]_\varepsilon . \end{aligned} \qquad (8)$$

Here, the square brackets represent the inner product. The brackets with the σ index are valid on the boundary where the dynamic boundary conditions are set, and similarly the brackets with the ε index are valid on the boundary where the geometric boundary conditions are given. Quantities with a hat represent the known values on the boundaries.

For dynamic analysis, the expression $\left[\bar{q}, \bar{w}\right]$ in Eq. (8) corresponds to

$$\left[\bar{q}, \bar{w}\right] = \frac{1}{2}\rho h s^2 \left[\bar{w}, \bar{w}\right] . \qquad (9)$$

where s represents the Laplace argument, ρ represents the material density of the plate, and h is the thickness of the plate.

B. Mixed Finite Element Formulation

In order to derive the finite element formulation, first, the interpolation function must be chosen. Since only the first derivative of the variables exists in the functional, a linear shape function will satisfy the continuity and completeness requirements [15]. In this study, a four-noded linear rectangular master element is illustrated in Fig. 2 with a parent shape function,

$$\Psi_1 = \frac{1}{4}(1-\xi)(1-\eta), \quad \Psi_2 = \frac{1}{4}(1-\xi)(1+\eta), \quad \Psi_3 = \frac{1}{4}(1+\xi)(1-\eta), \quad \Psi_4 = \frac{1}{4}(1+\xi)(1+\eta), \tag{10}$$

expressed in the (ξ, η) coordinates that are a mere translation of (x, y).

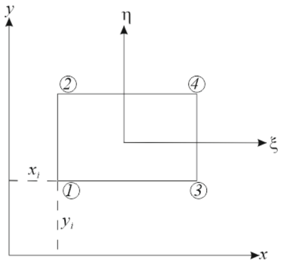

Figure 2. Four-noded rectangular master element.

All variables must be approximated by the given interpolation functions as follows:

$$\bar{w} = \sum_{i=1}^{4} \bar{w}_i \Psi_i(\xi,\eta)$$
$$\bar{M}_x = \sum_{i=1}^{4} \bar{M}_{x_i} \Psi_i(\xi,\eta)$$
$$\bar{M}_y = \sum_{i=1}^{4} \bar{M}_{y_i} \Psi_i(\xi,\eta) \tag{11}$$
$$\bar{M}_{xy} = \sum_{i=1}^{4} \bar{M}_{xy_i} \Psi_i(\xi,\eta)$$
$$h = \sum_{i=1}^{4} h_i \Psi_i(\xi,\eta).$$

and they are inserted into Eq. (8). After simplifying the functional with respect to nodal variables, the element matrix of the viscoelastic plate is derived.

III. Illustrative Examples and Discussion

A. Example 1

In this example, the dynamic response of a simply supported viscoelastic plate of uniformly varying cross sections, as illustrated in Fig. 3, is considered.

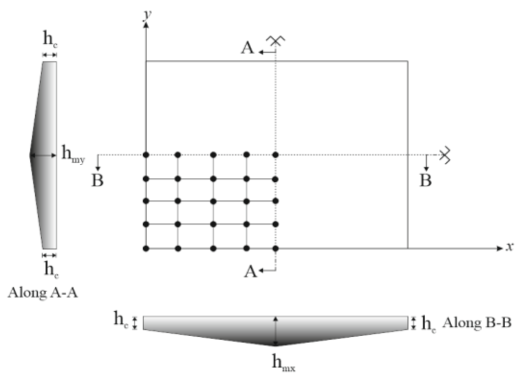

Figure 3. Geometric properties of the simply supported rectangular plate of uniformly varying cross sections.

Due to the symmetry, computations are carried out for a quarter of the plate using 4 × 4 mesh sizes as in Fig. 3. To model the behavior of a viscoelastic plate material, all numerical examples are employed for the four-parameter model presented in Fig. 4.

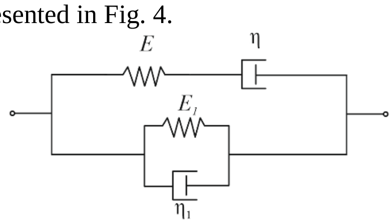

Figure 4. Mechanical model of the four-parameter model.

In order to determine the frequency of vibration, a free vibration analysis is performed. While keeping the thickness at the corner sides of the plate constant at h_c = 0.1 m, three uniformly varying thickness problems are solved:

- *Case 1*. A plate with a constant thickness along the x- and y-axes; $h_{mx} = h_{my} = h_c$.
- *Case 2*. At the center of the plate along the x-axis; $h_{mx} = 2h_c$, $h_{my} = h_c$.
- *Case 3*. At the center of the plate along the x-axis; $h_{mx} = 4h_c$, $h_{my} = h_c$.

The effects of the variation of the height of the cross section on the time-dependent displacement values at the center of the plate subjected to a step load q_0 = 10 kPa are computed for the following material and geometric properties. The material properties are $E = E_1 = 3 \times 10^7$ kPa, $\eta = \eta_1$ = 1,500 kPa s, and Poisson's ratio ν = 0.3. The material density of the plate, ρ, is assumed to be 2,000 kg/m^3. The geometric properties are length = width = 4 m. The results are presented in Fig. 5 by employing Durbin's inverse Laplace transform technique for aT = 5, N = 200, and T = 0.2 s. Where, T is the solution interval, a is the constant in numerical inverse Laplace transform, and N is the total number of terms of interest in the time interval. A viscoelastic plate vibrates in the same period as an elastic plate for small values of viscosity coefficient. Theoretical validation has already been satisfied by [11] for viscoelastic plates modeled by a Kelvin solid model. Moreover, it is observed that there is an inverse proportion between the vibration period and the thickness of the plate. If the thickness of the plate along B-B is increased, the vibration period of the plate decreases.

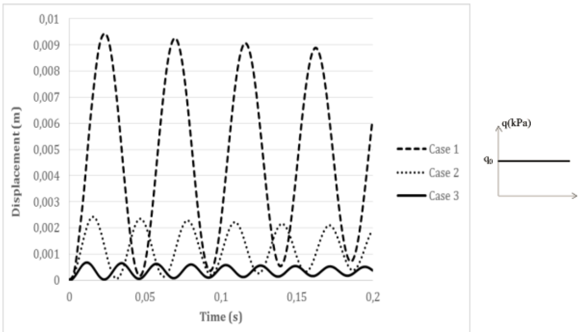

Figure 5. Effect of variation of the height of the cross section along B-B on the dynamic displacement–time variation results.

B. Example 2

This problem is solved for the four-parameter model (see Fig. 4) employing different η/E and η_1/E_1 ratios in order to show the damping effect on the dynamic time-dependent displacement values. For the analysis, plates with a constant thickness along the *x*- and *y*-axes (h_{mx} = h_{my} = h_c = 0.1 m) are considered. The time-dependent displacement values at the center of the plate subjected to a rectangular impulsive load q_0 = 10 kPa and t_0 = 1 s are computed for the following material and geometric properties. The material properties are $E = E_1 = 3 \times 10^5$ kPa and Poisson's ratio ν = 0.3. The material density of the plate, ρ, is assumed to be 2,000 kg/m^3. The geometric properties are length = width = 4 m. The results are presented in Fig. 6 by employing Durbin's inverse Laplace transform technique for aT = 5, N = 200, and T = 4 s for two different η/E and η_1/E_1 ratios: $\eta/E = \eta_1/E_1$ = 0.01 and $\eta/E = \eta_1/E_1$ = 0.001.

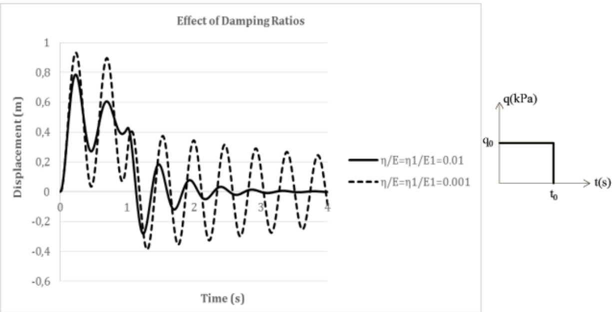

Figure 6. Effect of different η/E and η_1/E_1 ratios.

As expected, increasing the damping ratio causes the response to reach the static response much faster. The dynamic behavior of the plates dies out with time and approaches the static state.

Moreover, in this example, the effect of the variation of the height of the cross section along A-A and B-B on the vibration behavior of the viscoelastic plate is also considered. The results are presented for $\eta/E = \eta_1/E_1$ = 0.01. While keeping the thickness at the supported sides of the plate constant at h_c = 0.1 m, three uniformly varying thickness problems are solved:

- *Case A.* At the center of the plate along the *x*- and *y*-axes; $h_{mx} = h_{my} = 2h_c$.
- *Case B.* At the center of the plate along the *x*- and *y*-axes; $h_{mx} = h_{my} = 4h_c$.
- *Case C.* At the center of the plate along the *x*-axis; $h_{mx} = 4h_c$, $h_{my} = h_c$.

The time-dependent displacement values at the center of the plate with variable thicknesses are computed and presented in Fig. 7.

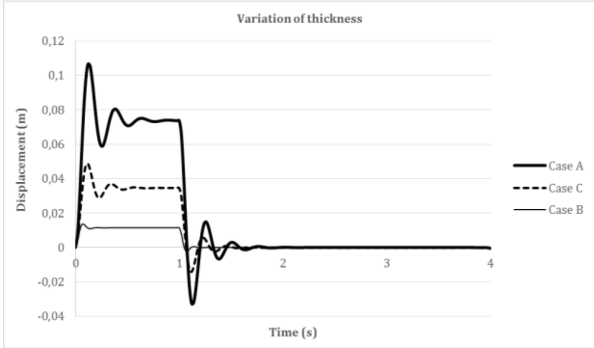

Figure 7. Effect of variation of the height of the cross section along A-A and B-B on the displacement–time variation results.

As anticipated, an increase in the value of plate thickness at the center of the plate exerts a decrease in the central displacement response. Increasing the thickness at the center of the plate along both the *x*- and *y*-axes causes the dynamic behavior of the viscoelastic plate to disappear much faster.

IV. CONCLUSION

In this paper, a functional with the necessary boundary condition terms for the free vibration analysis of viscoelastic classical thin plates of uniformly varying thicknesses is constructed using a systematic procedure based on the Gâteaux differential approach. A new mixed-type finite element composed of four nodes, each with four degrees of freedom, is formulated for this analysis. By the dynamic analysis of square viscoelastic plates with variable cross sections (Examples 1 and 2), the vibration behavior of a simply supported viscoelastic plate modeled with a four-parameter model is analyzed for small values of viscosity coefficient. In addition, by Example 2, the influence of the damping ratio (η/E and η_1/E_1 ratios) on the dynamic response is investigated, since this ratio is one of the most important factors affecting the viscoelastic behaviors. It is thought that the results of this paper may serve as a benchmark and may be useful for scientists and researchers in assisting them in evaluating the accuracy of their further investigation results for comparison purposes. The same approach can be applied for higher-order plate theories as well as shell theories. Following the described methodology, some of these problems are under study.

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