DQM Application on Classical Problem "Stagnation Point Flow"

Cemre Melike Pehlivan[1], Melih Fidanoğlu[1], Güven Kömürgöz[2], and İbrahim Özkol[1]
[1]Faculty of Aeronautics and Astronautics ITU, Istanbul, Turkey
[2]Faculty of Electrical and Electronical Engineering, ITU, Istanbul Turkey
Email: {Pehlivance, fidanoglum, komurgoz, ozkol}@itu.edu.tr

***Abstract*—In this study, a classical problem stagnation point flow was revisited by using powerfull enough semi numerical analytical technique DQM. A general expression of stagnation point flow is obtained. This equation solved with differential quadrature method (DQM) and Galerkin method. It is seen that obtained results are very closed to each other. But using DQM is more practical and faster than Galerkin. Because polynomial fitting in Galerkin method requires much more time than DQM**

***Index Terms*—component; stagnation point flow, differential quadrature method, galerkin method**

I. Introduction

In recent years, the study of stagnation point flow has gained a lot of importance because stagnation point flow that is ubiquitous and involves interaction of several physical problems. Stagnation point flow is an important phenomena since all interactions between solid structures and fluid flow involve stagnation point or lines. Knowing the rate of change of the physical variables around the surroundings of the flow is very important in terms of engineering. Velocity decrease but highest heat transfer and highest pressure occur around the surroundings of the flow. According to this, the shape of the structure or selection of material may change. Some sectors that play an important role in the stagnation point flow are electronic, hydrodynamic and aerodynamic [1]. Many flow and heat transfer problems such as microelectronics cooling design, heat transfer in atmospheric reentry, heat exchanger, drag reduction, prediction of skin friction problems that are encountered in engineering applications are stagnation point flow nature [2].

In both theory and practice, the analysis of stagnation point flow is very important. Hiemenz (1911) first examined the two-dimensional flow of a fluid toward a fixed plane wall. Hiemenz demonstrated that Navier Stokes equations could be reduced to third order nonlinear ordinary differential equations as obtained in stagnation point flow section. Then the stagnation point flow has been successfully applied to the numerous problems that include different physical conditions.

In this study, first governing equations of the stagnation point flow are obtained and the equation is solved by two different numerical techniques. One of them is differential quadrature method that is very powerful technique and the other one is Galerkin method. In conclusion, each numerical method results are compared according to their results which show that results obtained from DQM and Galerkin are very close to each other.

II. Mathematical Modelling of Stagnation Point Flow

The geometry of stagnation point flow equation is shown in Fig. 1. Initially this problem was studied by Hiemenz (1911). Hiemenz investigated that thanks to modification of the potential flow solution and by using similarity solution that decrease the number of variables, stagnation point flow can be analyzed by Navier Stokes equations [3].

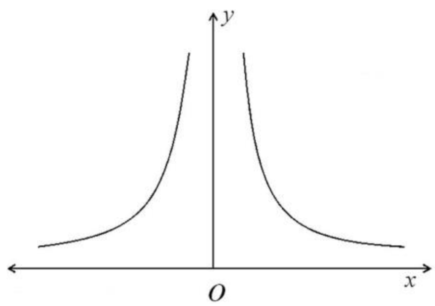

Figure 1. The geometry of stagnation point flow

Complex potential function near stagnation point is as in equation (2.1),

$$F(z) = Uz^2 \tag{2.1}$$

where $z = x + iy$, U is magnitude of the velocity. If z is substituted in the equation (2.1), the equation is rearranged and the stream function and velocity potentials can be extracted from equation as seen in equation (2.2) and (2.3).

$$\psi = 2Uxy \tag{2.2}$$

$$\phi = U(x^2 - y^2) \tag{2.3}$$

Correspondingly velocity components are calculated in equation (2.4) and (2.5)

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} = 2Ux \tag{2.4}$$

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$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} = -2Uy \qquad (2.5)$$

For an inviscid, incompressible and steady flow along the same streamline, *Bernoulli's Equation* given in equation (2.7),

$$p_0 + V_0 = p + \frac{1}{2}\rho V^2 \qquad (2.7)$$

where p_0 and V_0 are pressure and velocity at stagnation point, and ρ is the density. At the stagnation point $V_0 = 0$. If velocity is substituted into equation (2.7), distribution of the pressure is found as in equation (2.8).

$$p = p_0 - 2\rho U^2 (x^2 + y^2) \qquad (2.8)$$

From the above mentioned equation pressure distribution and velocity, satisfy both potential flow problem and equations of motion for a viscous, incompressible fluid exactly. But for potential flow, viscous shear terms in the Navier Stokes equations are zero. In order to satisfy no-slip boundary condition, viscous shear terms are required. For this reason Hiemenz attempted to modify potential flow theory that provide both satisfy the equation of motion for a viscous, incompressible fluid and no slip boundary condition.

For this reason x component of the velocity is taken as equation (2.9) to satisfy the desired conditions.

$$u = 2Uxf'(y) \qquad (2.9)$$

The prime denotes the gradient with respect to y. By using continuity equation vertical component of the velocity is found as in equation (2.10)

$$v = -2Uf(y) \qquad (2.10)$$

Thus continuity equation is satisfied by velocity fields for all functions f(y), thanks to modification in the potential flow field that is done by Hiemenz. If we lay down as a condition that $f(y) \to y$ as $y \to \infty$, the potential flow solution is valid far from the boundary.

Further restrictions on the function f are done by following Navier Stokes equations in equation (2.11) and (2.12).

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + v_{,}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad (2.11)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + v_{,}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \qquad (2.12)$$

It should be noted that $v_{,}$ with subscript represents the kinematic viscosity that is equal to the $v_{,} = \dfrac{\mu}{\rho}$ and v without subscript is the vertical velocity.

Expressions obtained foregoing for u and v are substituted into the equations (2.12) and the equation (2.13) is obtained.

$$4U^2 ff' = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2Uvf'' \qquad (2.13)$$

Through equation (2.13) pressure distribution is found as in equation (2.14).

$$P(x,y) = -2\rho U^2 f^2 - 2\rho Uvf' + g(x) \qquad (2.14)$$

If boundary condition $f(y) \to y$ for large values of y and equation (2.8) are used g(x) and pressure distribution are found in equation (2.15)

$$P(x,y) = p_0 - 2\rho U^2 f^2 + 2\rho Uv(1-f') - 2\rho U^2 x^2 \qquad (2.15)$$

From the equation (2.15), $\dfrac{\partial p}{\partial x}$ is found to substitute into the x momentum equation in (2.11) and equation (2.16) is found as

$$4U^2 x(f')^2 - 4U^2 xff'' = -\frac{1}{\rho}(-4\rho U^2 x) + v_{,}2Uxf''' \qquad (2.16)$$

After that equation (2.16) is rearranged, equation (2.17) is found.

$$\frac{v}{2U} f''' + ff'' - (f')^2 + 1 = 0 \qquad (2.17)$$

On the surface of the plate, the boundary condition $u(x,0) = 0$ requires that $f'(0) = 0$ and the boundary condition $v(x,0) = 0$ requires that $f(0) = 0$. In addition these boundary condition potential flow is $y \to \infty$ that is required $f(y) \to y$ or $f'(y) \to 1$ as $y \to \infty$. The boundary conditions are arranged as $f(0) = f'(0) = 0$ and $f'(y) \to 1$ as $y \to \infty$.

As a result, thanks to modification that is done by Hiemenz, potential flow satisfies both governing equations and viscous boundary conditions. Be able to solving the equations (2.17) for all kinematic viscosities and all flow velocities, eliminating $\dfrac{v}{2U}$ parameter from the equation is required. For this reason, the following change of variables is done with the equations (2.18) and (2.19).

$$\phi(\eta) = \sqrt{\frac{2U}{v}} f(y) \qquad (2.18)$$

$$\eta = \sqrt{\frac{2U}{v}} y \qquad (2.19)$$

The variable ϕ depends on the both η and y. Be able to expressed f''', f'', f' in terms of ϕ''', ϕ'', ϕ' we need to use chain rule for partial derivatives. According to this equations, f''', f'', f' are found as in terms of ϕ''', ϕ'', ϕ' and substituted into the equation the equation (2.20) is derived.

$$\phi''' + \phi\phi'' - (\phi')^2 + 1 = 0 \qquad (2.20)$$

And the boundary conditions are: $\phi(0) = \phi'(0) = 0$ and $\phi'(\eta) \to 1$ as $\eta \to \infty$.

III. SOLUTION METHODS

A. Differential Quadrature Method

The idea of differential quadrature method is first order derivative of a function with respect to a coordinate direction is approximated by a weighted linear sum of all values in the same domain and along same direction. Thanks to this approximation the differential equation is reduced to a set of algebraic equations. The most critical point of the DQM is computation of the weighting coefficients for the discretization of the first and second order derivatives.

Mathematical representation of DQM is in equation (3.1).

$$f_x(x_i) = \left.\frac{df}{dx}\right|_{x_i} = \sum_{j=1}^{N} a_{ij} f(x_j), \quad \text{for i=1,2,...,N} \qquad (3.1)$$

where $f_x(x_i)$ is the first order derivative of the function, N represent the number of grid points in the domain, j represents the grid point (i.e number of column), i represents the dimension of the problem (i.e number of row), a_{ij} is the weighting coefficients, x_j is the value of the grid points, $f(x_j)$ is the value of the function at different grid points. Also calculated weighting coefficients a_{ij} are different at different locations according to coordinate axis. The expressions that are used for the calculating the weighting coefficients are given in the equation (3.2), (3.3) and (3.4). [4]

$$M^{(1)}(x_i) = \prod_{k=1,k\neq i}^{N} (x_i - x_k) \qquad (3.2)$$

$$a_{ij} = \frac{M^{(1)}(x_i)}{(x_i - x_j)M^{(1)}(x_j)}, \quad \text{for i} \neq \text{j} \qquad (3.3)$$

$$a_{ii} = \frac{M^{(2)}(x_i)}{2M^{(1)}(x_j)} \qquad (3.4)$$

B. Galerkin Method

Through the method of weighted residual (WRM) a solution can be approximated analytically. A weighted residual method uses a finite number of functions. The method is a slight extension of that used for boundary value problems. The basic concept of the WRM is to drive a residual error to zero through a set of conditions. To obtain the approximate solution for the equation given in the differential form, approximation function is selected and is substituted to the differential equation. Result that is different than the zero is named as residual. This value that was obtained is multiplied by the specific weighted functions and the resulting product is tried to minimize In five steps, WRM can be applied to the problem

- The trial function is written by expanding unknown solution in a set of basis functions
- The trial function is satisfied the boundary conditions and initial conditions.
- Residual is defined.
- Weighted residual is set to zero and equations are solved.
- The error is examined by setting up successive approximations, and converge is shown the number of basis functions increase.

For the numerical solution of stagnation point flow equation, weighted residual method is applied by using Weierstrass theorem and trial function is defined as 8th order polynomial [5]. There are four main categories of weighted functions are selected such as subdomain method, collocation method, least squares method and Galerkin method. In this thesis stagnation point flow equation is solved by Galerkin method.

Use N trial functions for weight functions that is seen in equation (3.5)

$$w_n = f_n; \quad \text{n=1,2,3...,N} \qquad (3.5)$$

In the Galerkin method, the trial function is multiplied with the function and the integral over the region is taken as zero as seen from equation (3.6) [5].

$$\int_{\eta} w_n R d\eta = \int_{\eta} f_n R d\eta = 0; \quad \text{n=1,2,3,...,N} \qquad (3.6)$$

TABLE I. SOLUTION OF THE STAGNATION POINT FLOW EQUATIONS WITH DIFFERENTIAL QUADRATURE METHOD.

	DQM		
x	*f*	*f'*	*f''*
0	0	0	1.232591
0.5	0.133493	0.495688	0.758308
1	0.459161	0.777934	0.398013
1.5	0.887278	0.916461	0.176958
2	1.36193	0.973461	0.065825
2.5	1.85444	0.992659	0.020227
3	2.35256	0.998227	0.005078
3.5	2.85219	0.999806	0.001032
4	3.35212	0.999866	0.000169
4.5	3.85211	0.999661	0.000022
5	4.35211	1	0.000002

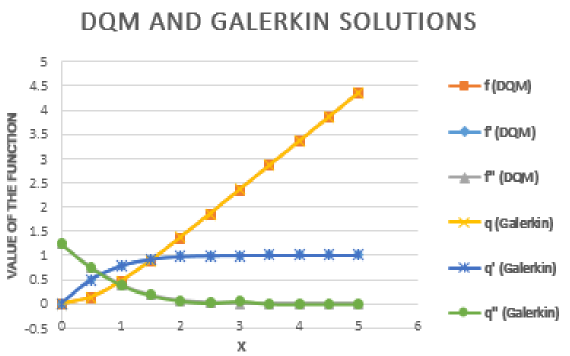

Figure 2. Solution of the stagnation point flow equations with differential quadrature method and Galerkin method.

IV. Results and Discussion

Stagnation point flow equation is solved using both differential quadrature method and Galerkin method. Obtained results are very close to each other for this reason graphics overlapped as seen in the Fig. 2. Also values of the functions at different locations according to both DQM and Galerkin are given in the Table I and Table II.

TABLE II. SOLUTION OF THE STAGNATION POINT FLOW EQUATIONS WITH GALERKIN METHOD.

	Galerkin Method		
x	***q***	***q'***	***q''***
0	0	0	1.246326
0.5	0.134173	0.495688	0.75537
1	0.460031	0.77793	0.397656
1.5	0.888205	0.91646	0.177526
2	1.363013	0.97346	0.065068
2.5	1.85548	0.99265	0.019598
3	2.353483	0.998227	0.056258
3.5	2.85308	0.999806	0.001296
4	3.35304	0.999867	-0.00068
4.5	3.8529	0.999661	0.000384
5	4.35283	1	-0.00094

As seen from values that are taken from the Table I and Table II, values are very close to each other. If two methods are compared, DQM is more preferable than the Galerkin method. Because polynomial curve fitting approach in Galerkin method requires more time than DQM.

V. Conclusion

In fluid mechanics, stagnation point occurs when the fluid impinges on a surface of the body. Further surroundings of the point is known as stagnation point flow. Many researchers put excessive emphasis on stagnation point flow that has effects on different kind of engineering disciplines such as drag reduction, heat and mass transfer near stagnation regions of bodies and so on. In this study stagnation point flow equation is solved by using differential quadrature and Galerkin method. Obtained results show that both of the methods give similar results. But using DQM is more practical and faster than Galerkin. Polynomial fitting in Galerkin method is taking much more time than DQM.

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