Algorithms and Software for Computing Inverse Distribution Functions in Power Equipment Based on Digital Twins

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***Abstract*—This research considers challenges and importance of accurately computing statistical distribution characteristics in the context of technical systems testing, particularly for fatigue and endurance tests of structural elements in complex systems like turbines and engines. These accurate computations are crucial for developing digital twins of equipment as traditional approximations often fall short in providing the necessary precision. Key distributions mentioned include the non-central student distribution, the distribution of the variation coefficient, and order statistics all of which are essential for justifying tolerance intervals and assessing the life characteristics of critical components. The article highlights the computational difficulties posed by infinite integration limits and the need for minimizing target functions, which complicate accurate calculations. To address these issues the article presents developed JavaScript algorithms and software designed to compute the required distribution characteristics for a broad range of continuous distributions lacking satisfactory approximations. These solutions aim to facilitate faster and more precise computations for engineering tasks. Additionally, the article discusses some of the most accurate approximations available for these distributions.**

***Keywords*—digital twin, technical condition of equipment, inverse distribution functions, student distribution**

I. Introduction

When developing diagnostic systems and predictive analytics for energy facilities [1] there is a problem of implementing digital twins of equipment providing the construction of virtual copies of real objects that function the same way as a physical object. The digital twin displays all the processes of a real piece of equipment in real-time mode [2]. The availability of a digital copy makes it possible to assess the condition of power equipment and predict the degradation phenomena in it [3]. In this case, an important role is played by statistical methods that ensure the computation of random variables. There are several random variables in applied problems of mathematical statistics and accurate computation of the variable’s distribution functions causes significant computational difficulties due to the presence of infinite integration limits, the need to minimize the target functions, the lack of satisfactory approximations etc. Such distributions include non-central student distribution, variation coefficient distribution, order statistics and others. Apart from problems of computing the power of statistical criteria values of quantiles of the non-central student distribution are necessary in reliability problems to justify the tolerance intervals and guaranteed life characteristics of technical systems [4, 5]. For example, application of the variation coefficient in engineering problems is considered in the paper [6]. Mathematical expectations and covariances of order statistics are widely used in applied problems of mathematical statistics related to the estimate of least squares with the least dispersion and in other fields [4–7]. Such as, Electrical Vehicle (EV) and EV charging applications [8–10].

The purpose of this paper is to develop algorithms and software for computing the inverse distribution functions. The algorithms under consideration are suitable for a wide class of continuous distributions inverse functions of which do not have any acceptable approximations. The software in question uses only the simplest and quite accurate approximations of standard distributions. The present paper contains approximations of the normal distribution, gamma function and incomplete gamma function [11]. The accuracy of which proved to be higher than known algorithms in Fortran [12]. The comparative basis was represented by WorksheetFunction objects built into Visual Basic for Applications (VBA) Excel. The authors offer JavaScript algorithms and software, which are available at http://inteh.mpei.ru/Cvar/. This language was selected due to its general accessibility and speed.

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II. Materials and Methods

The above problems are based on the distribution of the ratio of independent random variables [13, 14] $x = \zeta_1/\zeta_2$,

$$F(x) = \int_0^\infty F_1(t \cdot x) \cdot f_2(t) \cdot dt \qquad (1)$$

or

$$F(x) = 1 - \int_0^\infty f_1(t) \cdot F_2(\tfrac{t}{x}) \cdot dt \qquad (2)$$

where $f_1(t), F_1(t)$: density function and distribution function of a random variable; ζ_1, $f_2(t), F_2(t)$: density function and distribution function of a random variable ζ_2.

Accordingly, in n sample size from normal distribution $N(a,\sigma)$, with sample mean $\hat{a} = \bar{x} = \sum_{i=1}^{n} x_i/n$ and dispersion of $\hat{\sigma}^2 = s^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2/(n-1)$ the random variable $t' = (\bar{x} - a + \delta) \cdot \sqrt{n}/s$ has non-central student distribution with the distribution function:

$$F(t') = \beta = \int_0^\infty f(x) \cdot \Phi(t' \cdot x, \delta, 1) \cdot dx \qquad (3)$$

from where the quantile t'_β of level $\beta$is determined,

$$\Phi(x, a, \sigma) = \frac{1}{\sqrt{2 \cdot \pi} \cdot \sigma} \int_{-\infty}^{x} exp\left[-\frac{(t-a)^2}{2 \cdot \sigma^2}\right] \cdot dt \qquad (4)$$

normal law distribution function, δ : non-centrality parameter, $f = n-1$: number of freedom degree. Sampling distribution $\bar{x}$ is subject to normal law with parameters $N(a, \sigma/\sqrt{n})$. Variable $y = s^2 \cdot (n-1)/\sigma^2$ has χ^2 distribution with f degrees of freedom and density:

$$\phi(y) = \frac{2^{-f/2}}{\Gamma(f/2)} \cdot y^{f/2-1} \cdot exp(-y/2) \qquad (5)$$

where $\Gamma(x)$: gamma function (see module *stat.js*). The density of the sampling distribution of the standard deviation $x = \sqrt{s^2/f}$ in Eq. (3) is determined on the basis of theorem on the density of the random variable monotone function [13]:

$$f(x) = \phi(f \cdot x^2) \cdot (f \cdot x^2)' = \frac{2 \cdot \left(\frac{f}{2}\right)^{\frac{f}{2}}}{\Gamma\left(\frac{f}{2}\right)} \cdot x^{f-1} \cdot exp\left(-f \cdot \frac{x^2}{2}\right) \qquad (6)$$

Variation coefficient distribution $\gamma = \sigma/a$ is subject to non-central student distribution [15] with non-centrality parameter $\delta = \sqrt{n}/\gamma$ and number of degrees of freedom f, i.e., $\nu_\beta = \delta/t'_\beta = \sqrt{n}/(t'_\beta \cdot \gamma)$. Therefore, to compute quantiles of the variation coefficient, it is sufficient to have quantiles of the non-central student distribution. Direct computation of the distribution function of the sample variation coefficient is reduced to the equation:

$$F(v) = \beta = 1 - \int_0^\infty f(x) \cdot \Phi\left(x \cdot \frac{v}{\gamma}, 1, \gamma/\sqrt{n}\right) \cdot dx \qquad (7)$$

where the quantile ν_β of level β is determined, γ: general value of the variation coefficient. In this paper, both approaches were analyzed giving identical results in terms of accuracy and speed.

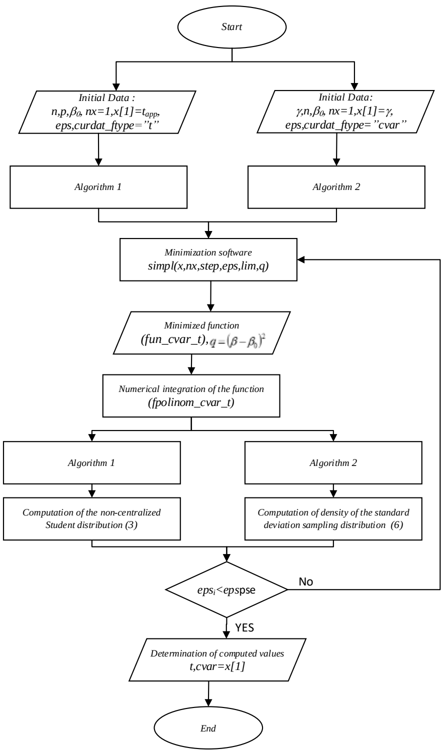

Fig. 1. Algorithms 1 and 2 for computing the inverse function of the non-central student distribution and the variation coefficient distribution. (n: sample size, p: set probability, δ: non-centrality parameter, 0: set probability, δ_0: computed probability, γ: general value of variation coefficient, nx = 1: number of variables in the minimization software *simpl*, *t,cvar*: computed value of inverse distribution function).

Thus, to compute the non-central student distribution functions and the variation coefficient, numerical integration of Eqs. (3) or (7) is required. Inverse distributions corresponding to the set probability were determined by successive approximations. For this purpose, the downhill (deformable polyhedron) simplex method was used in this paper [16].

Fig. 1 shows the algorithm for computing the inverse function of the non-central student distribution by integrating Eq. (3) with further minimization with a set accuracy of the quadratic function $q = (\beta - \beta_0)^2$ which is the square of the difference between the specified β_0 and computed β of confidence probabilities (we call this algorithm as the Algorithm 1).

Numerical integration of Eq. (3) was performed by the Bull method [17, 18] (*stat.js* module). *Integrate Function* parameters are: *k*: number of selected methods corresponding to the dimension of the polynomial approximant, 3, 4: Simpson method, 5: Bull method; *n step*: number of integration steps, *xl*, *xu*: lower and upper limits of integration, *f polynom*: name of integrating function, function returns the integral value. Function call: *beta = Integrate Function (k, n step, xl, xu, f polinom).* Supporting software is contained in *stat.js* module. In order to avoid possible errors, the software uses only the simplest and proven *Javascript* structure including data entry, displaying the results on the screen.

Similar algorithm is shown in Fig. 1 for distribution of the variation coefficient of variation (let's call this Algorithm 2), differing only by specifying the initial data and the integrable function (in this case Eq. (6)).

As computations showed, the fast and successful search for minimum points in *simpl* software significantly depends on initial approximations set at the input. Algorithm 1 uses the following normal approximation of quantile for this purpose [4, 5] (*invnontap* function in *stat.js* module):

$$t'(\beta,\delta,f) = \frac{\left(1-\frac{1}{4f}\right)\cdot\delta + z_\beta\cdot\sqrt{\left(1-\frac{1}{4f}\right)^2 - \frac{z_\beta^2}{2f} + \frac{\delta^2}{2f}}}{\left(1-\frac{1}{4f}\right)^2 - \frac{z_\beta^2}{2f}} \tag{8}$$

where δ is the non-centrality parameter, *f*: the number of degrees of freedom, z_β: quantile of the standard normal level distribution β, β, is the probability of confidence.

In terms of the variation coefficient distribution, an approximation [19] was used as a first order:

$$\frac{\nu_p}{\gamma} = \sqrt{\frac{\chi^2_{p,f}}{(1+1/\gamma^2)\cdot f - \chi^2_{p,f}\cdot f/(f+1)}} \tag{9}$$

where ν_p: quantile of the variation coefficient distribution of the probability level *p*, γ: general value of the variation coefficient, $\chi^2_{p,f}$: quantile of the chi-squared distribution.

The approximation Eq. (9) gives significant errors at values γ greater than 0.3.

In terms of programming, the problem of exact computation of characters (mathematical expectations $E(x_r)$ and dispersions $D(x_r)$) of order statistics in sample size *n* by forth integration [7–10] seems less complicated:

$$E(x_r) = \frac{\int_{-\infty}^{\infty}\varphi(x)\cdot x\cdot[1-F(x)]^{n-r}\cdot[F(x)]^{r-1}\cdot dx}{B(r,n-r+1)} \tag{10}$$

$$ED(x_r) = \frac{\int_{-\infty}^{\infty}\varphi(x)\cdot x^2\cdot[1-F(x)]^{n-r}\cdot[F(x)]^{r-1}\cdot dx}{B(r,n-r+1)} - E^2(x_r) \tag{11}$$

where $r = 1,\dots n,$: number of order statistics, $\varphi(x), F(x)$: density and function of distribution, $B(a,b) = \Gamma(a)\cdot\Gamma(b)/\Gamma(a+b)$: beta function. Numerical integration is performed using the *dataorder* function in *cvar.js* module.

III. Result and Discussion

Table I shows results of computing quantiles of the non-central student distribution based on Algorithm 1. The software showed varied testing size *n* of 3 to 50, probability *p* of 0.01 to 0.99 and confidence probabilities *β:0*.9, 0.95 and 0.99. Table I shows fragment of computations for *n* of 3 to 10. Computation time in the full range of all parameters is no more than 10–15 seconds on PC with low performance (performance index −1.0, RAM–4 Gb, processor – Intel® Pentium ® 1.9 GHz). Computational accuracy is nearly 10^{-5}.

To compare computational accuracy of the non-central student distribution function (see Table II) we used table values and the algorithm [20] (*prncst* function in *stat.js* module) as well as the approximation [11] (*sf54r* function in *stat.js* module). As can be seen from Table II the accuracy is very high in all options. Discrepancies are observed only in the sixth to seventh decimal place.

In software under consideration, the upper limit of integration was set to 5 but can be changed by the user depending on a required accuracy of computations.

Table III presents result for computing quantiles of relative variation coefficients as per algorithm 2 υ_p/γ for sample sizes *n* = 3–10. General variation coefficients 0.05, 0.3 and 0.5 and probabilities *p* ranging from 0.01 to 0.99. Speed and accuracy are almost equivalent to the previous example. Table IV shows similar computations but performed using the approximation Eq. (8). A comparison of Tables III and IV confirms the well-known result [18] that the approximation Eq. (8) gives an error of no more than 3–5% at values *γ* in the range of 0.05–0.3 but at larger values of the variation coefficient it increases to 9–15% depending on the value *n*. Applying Algorithm 1 to compute the percentage points of the variation coefficient gives exactly the same results as Algorithm 2. Therefore, it is recommended to use Algorithms 1 or 2 for all practical values of sample sizes *n*, values *γ and probabilities p for exact and verification computations a*nd Algorithm 1 combined with approximations [11, 20] for fast computations.

TABLE I. QUANTILES OF THE NON-CENTRAL STUDENT DISTRIBUTION FOR SAMPLE SIZES n = 3–10, (NON-CENTRALITY PARAMETER $\delta = z_p \cdot \sqrt{n}$)

β	n	p = 0.01	0.05	0.1	0.3	0.5	0.7	0.9	0.95	0.99
β = 0.99	3	−1.3536792	−0.5105654	0.1245945	2.8786357	6.9645544	13.0712125	24.2407483	30.0860653	41.3883239
	4	−1.8472004	−0.8857286	−0.2451913	1.8985941	4.5407048	8.223549	14.7597818	18.1669008	24.7745658
	5	−2.296835	−1.214473	−0.5315038	1.4986883	3.7469478	6.7518861	11.9891732	14.7096064	19.9882608
	6	−2.7131195	−1.5144931	−0.7810481	1.2576921	3.3649308	6.1001286	10.8048983	13.2408281	17.9659398
	7	−3.1028759	−1.7934666	−1.0080186	1.0830341	3.1426678	5.7580231	10.210297	12.5087421	16.9644092
	8	−3.4708345	−2.0558281	−1.2188556	0.9431022	2.9979522	5.5626035	9.8915944	12.1205258	16.43825
	9	−3.8204406	−2.3045162	−1.4171848	0.8241254	2.8964601	5.4469441	9.7212346	11.9167803	16.166635
	10	−4.1543004	−2.5416339	−1.6053317	0.7190867	2.8214376	5.3788426	9.6383305	11.8215914	16.0445284
β = 0.95	3	−1.9566387	−1.1070312	−0.5793325	1.0051572	2.919986	5.6762818	10.6612588	13.2604081	18.2778621
	4	−2.4923255	−1.4866088	−0.8877785	0.7109377	2.3533635	4.5308156	8.3238654	10.2877502	14.0847225
	5	−2.9760095	−1.8286089	−1.1600269	0.5300989	2.1318469	4.1607664	7.6174653	9.3974797	12.8374608
	6	−3.4205601	−2.1427448	−1.4080225	0.3909637	2.0150488	4.0125171	7.3637937	9.0819335	12.3992868
	7	−3.8343448	−2.4350796	−1.6378358	0.2733724	1.9431808	3.9544098	7.2901881	8.9941472	12.2808368
	8	−4.2231165	−2.7097237	−1.8532134	0.169149	1.8945787	3.9403276	7.3027406	9.0150293	12.3145682
	9	−4.5910212	−2.9696243	−2.0567122	0.0742025	1.8595475	3.9502649	7.3612672	9.093709	12.4290667
	10	−4.9411662	−3.2169811	−2.2501852	−0.0138155	1.8331132	3.974285	7.4460279	9.205277	12.5893981
β = 0.90	3	−2.3570178	−1.4550122	−0.9262696	0.4518083	1.8856184	3.8590338	7.3753589	9.1997526	12.7140114
	4	−2.9103929	−1.844764	−1.2341415	0.2596394	1.6377444	3.3860116	6.3756891	7.9131314	10.8764665
	5	−3.4089776	−2.196217	−1.5099056	0.1169958	1.5332068	3.2551829	6.1320779	7.6022592	10.4334531
	6	−3.8660398	−2.5186127	−1.7621781	−0.0031124	1.4758845	3.2272904	6.1082676	7.5735243	10.3920429
	7	−4.2904886	−2.8181408	−1.996231	−0.1097249	1.4397552	3.2413693	6.1716037	7.6562707	10.5089838
	8	−4.6884925	−3.0990964	−2.215594	−0.2070261	1.414924	3.2754144	6.2751304	7.7902943	10.6986604
	9	−5.0645002	−3.3645879	−2.4227766	−0.2973447	1.3968153	3.3197823	6.3986238	7.9497059	10.9243219
	10	−5.4218393	−3.6169423	−2.6196388	−0.3821411	1.3830284	3.369699	6.5322147	8.1219114	11.1680831

TABLE II. INTEGRAL OF THE NON-CENTRAL STUDENT DISTRIBUTION PROBABILITIES FOR NUMBERS OF DEGREES OF FREEDOM F AND THE NON-CENTRALITY PARAMETER δ

t'	f	δ	$F(t')_{tabl.}$	$F(t')$ Algorithm 1	$F(t')$ [4, 17]
3	1	0	0.8975836	0.8975836	0.8975835
3	2	0	0.9522670	0.9522670	0.952267
3	3	0	0.9711656	0.9711656	0.9711654
3	1	0.5	0.8231218	0.8231219	0.8231218
3	2	0.5	0.9049021	0.9049022	0.9049021
3	3	0.5	0.9363471	0.9363472	0.9363471
3	1	1	0.7301025	0.7301026	0.7301025
3	2	1	0.8335593	0.8335594	0.8335594
3	3	1	0.8774009	0.8774010	0.8774009
3	1	2	0.5248571	0.5248572	0.524857
3	2	2	0.6293851	0.6293857	0.6293856
3	3	2	0.6800267	0.6800272	0.6800271
3	1	4	0.2059011	0.2059013	0.2058895
3	2	4	0.2112146	0.2112149	0.2112149
3	3	4	0.2074727	0.2074731	0.2074709
15	15	7	0.9981123	0.9981130	0.9981129
15	20	7	0.9994867	0.9994874	0.9994873
15	25	7	0.9998384	0.9998392	0.9998391
0.05	1	1	0.1686106	0.1686106	0.1686106
0.05	2	1	0.1696795	0.1696795	0.1696795
0.05	3	1	0.1701041	0.1701041	0.1701041
4	10	2	0.9247682	0.9247683	0.9247682
4	10	3	0.7483138	0.7483139	0.7483139
4	10	4	0.4659800	0.4659802	0.4659802
5	10	2	0.9761868	0.9761873	0.9761872
5	10	3	0.8979688	0.8979689	0.8979689
5	10	4	0.7181899	0.7181905	0.7181904
6	10	2	0.9923652	0.9923659	0.9923658
6	10	3	0.9610338	0.9610342	0.9610341
6	10	4	0.8688001	0.8688007	0.8688007

TABLE III. QUANTILES OF SAMPLE VARIATION COEFFICIENTS v_p/γ FOR SAMPLE SIZES n = 3–10, COEFFICIENTS OF VARIATION AND PROBABILITIES p, COMPUTED BY ALGORITHM 2

	n	***p* = 0.01**	**0.05**	**0.1**	**0.3**	**0.5**	**0.7**	**0.9**	**0.95**	**0.99**
γ = 0.05	3	0.1002102	0.2263956	0.3244862	0.5971515	0.8326885	1.0979006	1.51971	1.7344276	2.1533418
	4	0.1955307	0.3422886	0.4411568	0.6887522	0.8881656	1.1058458	1.4454562	1.616912	1.9503398
	5	0.2723549	0.421281	0.5154112	0.7405768	0.9161456	1.1048769	1.3962501	1.5426097	1.8265554
	6	0.3327173	0.4783546	0.5672227	0.7744235	0.9329624	1.1017842	1.3606271	1.4901799	1.7410467
	7	0.3809654	0.521763	0.6057963	0.7985351	0.9441737	1.0982481	1.3332994	1.4506201	1.6774501
	8	0.4204085	0.5561038	0.6358691	0.816735	0.9521778	1.0947712	1.3114685	1.4193896	1.6277746
	9	0.4533285	0.5841021	0.6601271	0.8310509	0.9581766	1.0915088	1.2935028	1.3939159	1.587587
	10	0.4812971	0.6074751	0.6802103	0.8426637	0.9628401	1.0884974	1.278376	1.3726192	1.5542125
γ = 0.3	3	0.0988091	0.2234869	0.3208023	0.5945781	0.8371579	1.1205464	1.6048203	1.8737959	2.4638789
	4	0.1916642	0.3363389	0.4345316	0.6843881	0.8914472	1.1257984	1.5156774	1.7275832	2.1801959
	5	0.266242	0.413217	0.507012	0.7353899	0.9187356	1.122733	1.4564944	1.6353082	2.0107654
	6	0.3248317	0.468889	0.5577623	0.7688484	0.935102	1.1180122	1.4137562	1.5705593	1.8958781
	7	0.3717281	0.5113646	0.5956947	0.7927851	0.9459959	1.1131818	1.3810865	1.5220013	1.8117507
	8	0.4101458	0.5450793	0.6253817	0.8109232	0.9537644	1.1086514	1.3550862	1.4838882	1.7468857
	9	0.442285	0.5726575	0.649414	0.8252398	0.959582	1.1045128	1.3337646	1.4529646	1.6949849
	10	0.4696559	0.5957507	0.6693761	0.8368895	0.9641011	1.1007598	1.3158731	1.427234	1.6522852
γ = 0.5	3	0.0963902	0.2184256	0.3143173	0.5895348	0.844215	1.1621726	1.7900129	2.2137372	3.5256371
	4	0.1853166	0.3264466	0.4233633	0.6764026	0.8962664	1.1613484	1.6601829	1.9763649	2.8335787
	5	0.2564419	0.4000839	0.4931283	0.7261626	0.9223899	1.1540322	1.5763381	1.8337191	2.4843834
	6	0.312353	0.4536427	0.5422898	0.7590871	0.9380435	1.1461675	1.5170257	1.7369937	2.2692031
	7	0.3572225	0.4947248	0.5792782	0.7828211	0.9484576	1.138908	1.4724104	1.6662827	2.1213024
	8	0.3941057	0.527509	0.6084066	0.800924	0.9558808	1.132438	1.4373606	1.6118752	2.0123565
	9	0.4250757	0.5544631	0.6321201	0.8152957	0.9614377	1.1267084	1.4089261	1.5684337	1.9281692
	10	0.4515489	0.5771415	0.6519189	0.8270491	0.9657532	1.1216241	1.3852797	1.5327653	1.8607923

TABLE IV. QUANTILES OF SAMPLE VARIATION COEFFICIENT v_p/γ FOR SAMPLE SIZES n = 3–10, COEFFICIENTS OF VARIATION AND PROBABILITIES p, COMPUTED BY ALGORITHM 1

	n	***p* = 0.01**	**0.05**	**0.1**	**0.3**	**0.5**	**0.7**	**0.9**	**0.95**	**0.99**
γ = 0.05	3	0.1001271	0.2262073	0.3242163	0.5966545	0.8319954	1.0969864	1.518443	1.7329798	2.1515407
	4	0.1954087	0.342075	0.4408815	0.6883225	0.887611	1.1051551	1.4445527	1.6159006	1.9491183
	5	0.2722189	0.4210707	0.5151538	0.7402068	0.915688	1.1043252	1.3955525	1.5418387	1.8256421
	6	0.3325788	0.4781556	0.5669864	0.7741011	0.9325742	1.1013258	1.3600609	1.4895594	1.7403221
	7	0.3808293	0.521577	0.6055799	0.7982502	0.9438368	1.0978564	1.3328237	1.4501025	1.6768529
	8	0.4202771	0.5559299	0.6356706	0.8164799	0.9518803	1.0944298	1.3110596	1.4189464	1.627267
	9	0.4532026	0.58394	0.6599437	0.83082	0.9579108	1.0912061	1.2931441	1.3935295	1.5871483
	10	0.4811767	0.6073232	0.6800401	0.8424532	0.9625994	1.0882259	1.2780571	1.3722769	1.5538257
γ = 0.3	3	0.09605	0.2172356	0.3118091	0.5777352	0.8131055	1.0876403	1.5553379	1.8141354	2.3789943
	4	0.1876172	0.3292195	0.4253114	0.6697382	0.8721759	1.1011271	1.4815485	1.6880416	2.1284494
	5	0.2616875	0.4061352	0.4983089	0.7227038	0.9028091	1.1031524	1.4308339	1.6063538	1.9749015
	6	0.3201372	0.4621097	0.5496957	0.7577231	0.9215697	1.1018436	1.3933856	1.5480183	1.8690591
	7	0.3670601	0.5049526	0.5882339	0.7828952	0.9342463	1.099446	1.3643049	1.5037143	1.7906676
	8	0.4055792	0.5390324	0.6184626	0.8020256	0.9433887	1.0967322	1.3408851	1.4686065	1.7297164
	9	0.4378521	0.5669516	0.6429708	0.8171536	0.9502955	1.0939992	1.3215009	1.4399071	1.6806326
	10	0.4653681	0.5903558	0.6633498	0.8294779	0.9556982	1.0913649	1.3051119	1.4158821	1.6400424
γ = 0.5	3	0.0897277	0.2032663	0.2923857	0.5473457	0.7816551	1.0711509	1.6303835	1.9976299	3.0894911
	4	0.1754954	0.3090402	0.4006538	0.6393242	0.8459186	1.0938743	1.5572495	1.8494333	2.6440045
	5	0.2452276	0.3824969	0.4713567	0.6936726	0.8806008	1.1010176	1.5029189	1.7487529	2.3796072
	6	0.3005947	0.4365226	0.5217888	0.7302843	0.902416	1.1027923	1.4612321	1.675462	2.2035276
	7	0.345326	0.4782666	0.5600315	0.7569696	0.9174404	1.1023705	1.4281428	1.6193401	2.0769114
	8	0.3822714	0.5117491	0.5903089	0.7774682	0.9284317	1.1009802	1.401128	1.5747212	1.9808579
	9	0.413404	0.5393797	0.6150515	0.793816	0.9368278	1.0991624	1.3785656	1.5382161	1.9050832
	10	0.4400896	0.5626914	0.6357659	0.8072256	0.9434538	1.0971758	1.3593709	1.5076702	1.8435039

TABLE V. COMPARISON OF TABULATED, EQS. (10), (11) AND APPROXIMATE VALUES OF MATHEMATICAL EXPECTATIONS $E(X_R)$ AND DISPERSIONS $D(X_R)$ NORMAL ORDER STATISTICS IN THE SAMPLE SIZE n = 19

r	**$E(x_r)_{tabl.}$**	**$D(x_r)_{tabl.}$**	**$E(x_r)_{integr.}$**	**$D(x_r)_{integr.}$**	**$E(x_r)_{appr.}$**	**$D(x_r)_{appr.}$**
10	0.0000000	0.0807910	0.0000000	0.0807910	0.0000000	0.0807687
11	0.1307249	0.0812876	0.1307249	0.0812876	0.1307244	0.0812652
12	0.2637429	0.0828340	0.2637429	0.0828340	0.2637420	0.0828110
13	0.4016423	0.0856173	0.4016423	0.0856173	0.4016411	0.0855933
14	0.5477074	0.0900219	0.5477074	0.0900219	0.5477064	0.0899966
15	0.7066115	0.0967945	0.7066115	0.0967945	0.7066120	0.0967685
16	0.8858619	0.1074740	0.8858620	0.1074741	0.8858683	0.1074515
17	1.0994531	0.1257139	1.0994531	0.1257139	1.0994823	0.1257186
18	1.3799385	0.1627856	1.3799385	0.1627857	1.3800917	0.1630099
19	1.8444815	0.2799358	1.8444815	0.2799358	1.8461875	0.2843166

Table V shows the results of computing the mathematical expectations and dispersions of the normal order statistics for a test volume of 19. The same Table V presents for comparison tabulated [7] and approximate values obtained by David-Johnson approximation (*ordern* function, *stat.js* module) with the series expansion procedure $(n+2)^{-3}$. *Ordern* function contains derivatives up to the 6th distribution function as well as comments. As can be seen from the table discrepancies are found only in 4–5 digits after the decimal point. The speed is too high to be analyzed.

Forth integration of distribution functions proposed in this paper also allows us to compute characters of order statistics for distribution laws other than normal. Weibull distribution function for example with b and c parameters is as follows:

$$F(x) = 1 - exp\left(-\frac{x}{c}\right)^{b}, \quad (12)$$

By reducing the distribution to the form with shift $a_w = ln\, c$ and scale parameters $\sigma_w = 1/b$ $z = (ln\, x - a_w)/\sigma_w$, we obtain ratios for the function and density of the distribution:

$$F(z) = 1 - e^{-e^{z}}; \varphi(z) = \frac{\partial F(z)}{\partial z} = z \cdot e^{z-e^{z}}, \quad (13)$$

which are put into Eqs. (10) and (11). Table VI presents comparative computations of characters for order statistics of the Weibull distribution by integrals Eqs. (10) and (11) (*dataorder* function in *cvar.js* module) and approximations based on David-Johnson approximation (*orderw* function, *stat.js* module). As can be seen from Table VI, the approximation accuracy is quite high (differences are observed in the third-fourth decimal place). Users are advised to make their own decision on the use of one or another model depending on the required accuracy of computations. The proposed approximation also enables to compute covariances of order statistics. An accurate solution for covariances is not considered since it requires computing double integrals. Comparisons can be made over extensive tables [7]. Note that since the Weibull distribution is not symmetric the table shows all order statistics from 1 to n unlike the normal law.

TABLE VI. COMPARISON OF EQS. (10), (11) AND APPROXIMATE VALUES OF MATHEMATICAL EXPECTATIONS $E(X_R)$ AND DISPERSIONS $D(X_R)$ NORMALIZED WEIBULL ORDER STATISTICS IN THE SAMPLE SIZE n = 19

r	$E(x_r)_{integr.}$	$D(x_r)_{integr.}$	$E(x_r)_{appr.}$	$D(x_r)_{appr.}$
1	−3.5215617	1.6440953	−3.5271735	1.6405747
2	−2.4943774	0.6451776	−2.4946879	0.642366
3	−1.9657835	0.3954499	−1.9658067	0.3943384
4	−1.6023936	0.284645	−1.6023818	0.2840921
5	−1.3207016	0.2224915	−1.3206864	0.2221704
6	−1.0871679	0.1828865	−1.0871545	0.1826796
7	−0.8848755	0.1555627	−0.8848646	0.1554195
8	−0.7039936	0.1356816	−0.703985	0.1355771
9	−0.5381864	0.1206753	−0.5381797	0.120596
10	−0.3829977	0.1090657	−0.3829925	0.1090038
11	−0.2350141	0.0999558	−0.2350102	0.0999065
12	−0.0913642	0.0927892	−0.0913612	0.0927496
13	0.0506476	0.0872332	0.0506501	0.0872014
14	0.1938974	0.0831299	0.1938999	0.0831051
15	0.3419289	0.0805034	0.3419325	0.0804863
16	0.4998497	0.0796559	0.4998575	0.0796503
17	0.6764606	0.0815063	0.6764831	0.0815288
18	0.8909808	0.0889711	0.8910712	0.0891146
19	1.2076475	0.116379	1.2083904	0.1179448

Comparative computations in tables showed the high accuracy of the computations and features of the existing approximations. For example, the algorithms for approximating the non-central student distribution function give very accurate approximations over the entire range of probabilities, sample sizes and non-central parameters and are quite suitable for subsequent minimization to obtain percentage points. At the same time the well-known McKay approximation for the coefficient of variation gives errors of up to 15% for the variation coefficient values greater than 0.3.

Modified Javascript software and approximations presented in full and readily accessible to let us easily extend the obtained solutions to other statistical problems related to the exact distribution of the ratios of two independent random variables.

Developed software for integrating functions with infinite limits to obtain characters of order statistics to enable necessary computations for a large class of continuous distribution functions (normalized normal and Weibull distribution are considered in the paper) and presented approximations by David-Johnson method with the series expansion procedure. The computations presented in tables showed a fairly high expansion accuracy compared to the integral solutions (discrepancies are observed in the third or fourth decimal place). Users are advised to make their own decision on the use of one or another model depending on the required accuracy of computations.

IV. CONCLUSION

The developed algorithms and software for computing the inverse functions of the non-central student distribution and the distribution of the coefficient of variation based on a combination of numerical integration and the simplex minimization method allow to obtain fast and accurate solutions for the application of these functions in problems of applied statistics. Implementation of the proposed solutions will ensure the creation of digital twins of power equipment most fully reproducing the behavior of a real object in real time.

CONFLICT OF INTEREST

The authors declare no conflict of interest.

AUTHOR CONTRIBUTIONS

Conceptualization, ALV and OMV; methodology, ALV; software, MMS; validation, MMS, CDA; formal analysis, ALV; investigation, MMS, ALV; resources, KMA;

writing—original draft preparation, ALV; writing—review and editing, MMS; visualization, CDA; supervision, ALV; project administration, OMV; funding acquisition, ALV; all authors had approved the final version.

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