



Research Paper

ESTIMATING THE ERROR OF FIELD VARIABLE FOR NUMERICAL APPROXIMATION TECHNIQUES USING MATLAB

Nitin Sawarkar^{1*}, Bhushan Mahajan¹, Manoj Baseshankar¹ and Dnyaneshwar Kawadkar¹

*Corresponding Author: Nitin Sawarkar, ✉ nitinsawarkar304@gmail.com

The various scientific laws, principles are used to develop mathematical models. Often the differential equations are used to describe the system. But, formulating the differential equations for most problems is difficult and hence obtaining the solutions by exact methods of analysis is a formidable task. The present research discusses the issue of the finding the field variable deviation for quadratic area of one dimensional continuum. The analysis provides the percentage error in the field variable for the selected trial functions. Approximate method is useful, if it is integrated with computer for the problem involving a number of complexities without making drastic assumption which otherwise complicated to attempt by classical methods. A genuine necessity for obtaining precise solution for the different numerical approximation methods is overcome by developing in-house computer program. Graphically results can be displayed to know the effect of considered weights and the constants assumed.

Keywords: Numerical methods, Field variable, MATLAB, Mathematical modeling

INTRODUCTION

The model problem considered is of an axially loaded bar having quadratic function of area. The unknown variable is the axial displacement of one dimensional continuum, $u(x)$ is attempted in the present research by means of numerical analysis technique where the basic inputs to a problem are known with arbitrary basic data. The models are often described in terms of algebraic, differential

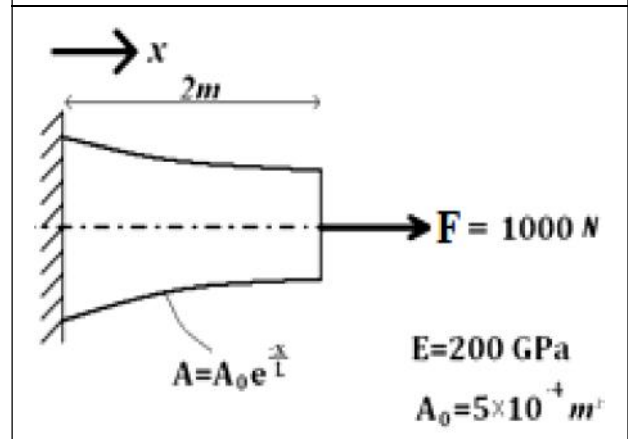
and integral equation which relates quantities of interest. Mathematical model is a set of relationship among variables that are expressed in essential features of physical system or the process in analytical terms. As a result, physical system or process can be described by means of mathematical model. The relationship that governs the system may take the form of algebraic, differential and integral equations.

¹ Mechanical Engineering Department, Bhausaheb Mulik College of Engineering, Nagpur, India.

The governing equations and boundary conditions are useful means to express the mathematical model. In general, engineering problems is often needs to be addressed considering the domain of interest. The differential equations are employed to illustrate the physical significance of the system under the domain of consideration. For such system, analytical solution can be found out using differential equation and boundary conditions. To establish mathematical model it is required to describe the relationships of primary and secondary variables. The solution characteristics can be expressed in the algebraic form. For the well defined simple engineering problems, there are standard known analytical techniques available to find solution. But, there are many engineering problems for which exact solution cannot be easily obtained using analytical techniques. The failure to obtain an exact solution may be attributed to either the complex governing differential equation or the difficulties in dealing with the boundary and initial conditions. To deal with such problems, numerical approximation is best alternative. An approximate technique may become the effective tool to define the nature of unknown variable and enable to determine the solution at discrete points in the problem domain.

In the present paper the numerical techniques like Direct Integration, Method of Weighted Residuals, Ritz Method and FEM are used to determine the deviation in the field variable at the selected node position. For this purpose the one dimensional continuum is selected as shown in Figure 1. The computer programming technique is employed to minimize computational cost and time. The

Figure 1: Axially Loaded 1D Continuum

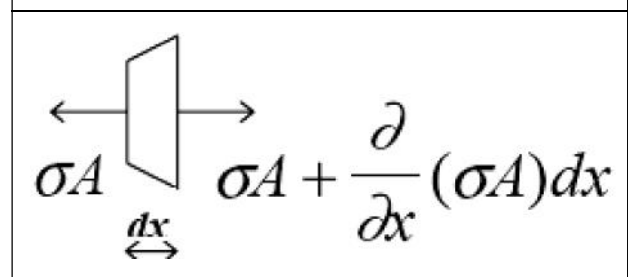


developed computer program uses MATLAB tool and developed sets of subroutines and functions. The programs are intended as a primary component for generalization to obtain the results at any position on the continuum. The applications can be extended for variety of similar complex problems.

DEVELOPMENT OF GOVERNING EQUATIONS

Consider an axially loaded 1D bar having varying area A with one end fixed at $x=0$ and externally applied force F at $x=L$ as shown in Figure 1. The Free Body Diagram (FBD) is shown in Figure 2. The axially loaded continuum considered here has length $L=2\text{ m}$ and area as exponential function of x given by $A = A_0 e^{\frac{-x}{L}}$, $E = 200\text{ GPa}$ and $F = 1\text{ KN}$.

Figure 2: Free Body Diagram



Governing equation is developed for 1D continuum having varying area. The unknown variable is the axial displacement given by $u(x)$.

From the Free Body Diagram (FBD) and using equilibrium equation,

$$\tau A + \frac{\partial}{\partial x}(\tau A)dx - \tau A = 0 \quad \dots(1)$$

where, A indicates the area which is assumed to be a continuous function with respect to x . The Equation (1) becomes,

$$\frac{\partial}{\partial x}(\tau A) = 0 \quad \dots(2)$$

From Hooke's Law, within elastic limit, Stress \propto Strain,

$$\text{or } \tau = E \frac{\partial u}{\partial x} \quad \dots(3)$$

where E is the modulus of elasticity of the system, Substituting τ in Equation (2)

The governing equation is,

$$\frac{\partial}{\partial x} = \left(EA \frac{\partial u}{\partial x} \right) = 0, 0 < x < L \quad \dots(4)$$

There are two boundary conditions associated with the problem,

The essential boundary conditions is.

$$u(x=0) = 0 \quad \dots(5)$$

and the natural boundary condition,

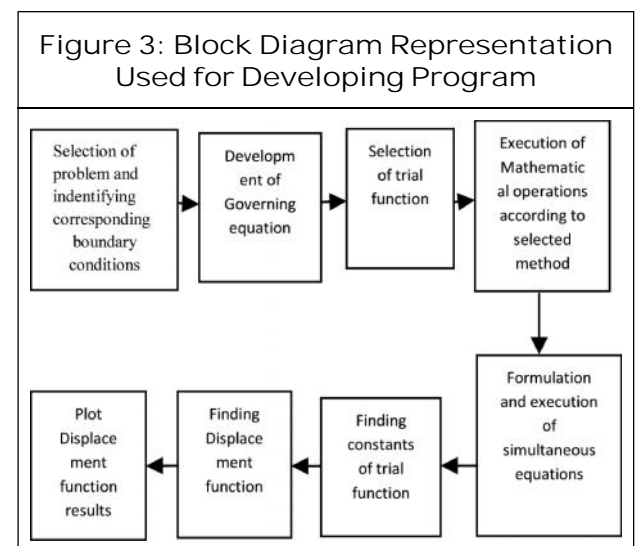
$$\frac{\partial u}{\partial x}(x=L) = \frac{F}{AE} \quad \dots(6)$$

where F is the axially applied load

DEVELOPMENT OF ROUTINE SCRIPT

MATLAB provides interactive platform for executing numerical computations. The

program code is written in MATLAB 7.10.0 (R 2010a) version. In-house computer program is developed different approximation numerical methods to determine the solutions to overcome long and tedious task which otherwise need to be attempted by hand calculation. The important steps which are used for writing the program script are indicated in the block diagram shown in Figure 3.



DIRECT INTEGRATION APPROACH

The direct integration technique is analytical way of finding exact solution (Pian and Tong, 1969). The result of approximate method is verified using classical methods, i.e., Direct Integration method. The technique is used for obtaining the result and finding the error of field variable at the discrete points.

Applying boundary condition, $u(x=0) = 0$

Substituting values of F , E and A at $x = 2$ and applying 2nd boundary condition

Equation (6) becomes

$$\frac{du}{dx} = \frac{F}{AE} \quad \dots(7)$$

Integrating the governing Equation (4),

$$\left(EA \frac{du}{dx}\right) = c_1 \Rightarrow \frac{du}{dx} = \frac{c_1}{EA} \quad \dots(8)$$

$$du = \frac{c_1}{E \times A_0 \times e^{-\frac{x}{L}}} dx \quad \dots(9)$$

Again Integrating,

$$u = \frac{1}{1/L} \frac{c_1 \times e^{\frac{x}{L}}}{EA_0} + C_2 \quad \dots(10)$$

Now, applying 2nd boundary condition to Equation (8), i.e., substitute $x = 2$, the result is $c_1 = 1000$

Put the value of c_1 in Equation (10) at $x = 0$,

$$c_2 = -2 \times 10^{-5}$$

Substituting the value of c_1 and c_2 in Equation (10), the generalized equation for displacement is,

$$u(x) = \frac{2 \times 1000 - 100}{200 \times 10^9 \times 5 \times 10^{-4}} - 2 \times 10^{-5} \quad \dots(11)$$

The displacement at interval of 0.25 m increment position from the starting point is

evaluated and depicted in Table 1 and the graphically results are represented in Figure 4.

Program Code for Direct Integration Method

The developed program calculate ' c_1 ' and ' c_2 ', the constants of the governing differential equation. The deformation is calculated in three statements of program. ' X ' is an array having different values of distances from fixed end. The second step call the value of X and final deformation values are obtained in the third step.

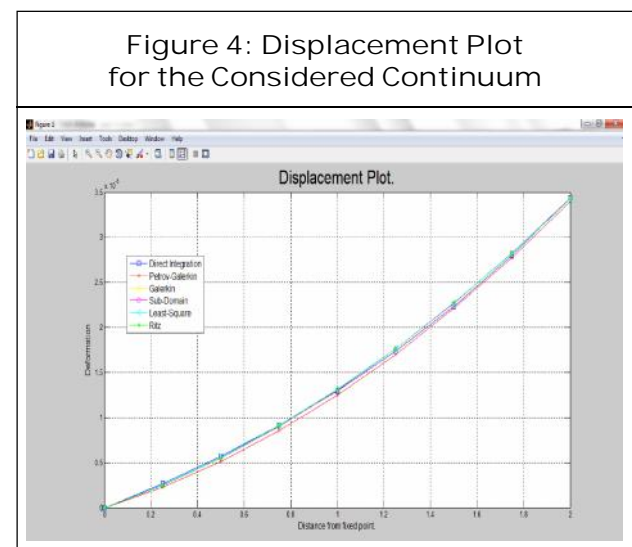


Table 1: Deformation Results for the Numerical Methods Considered

Methods Distance (m)	Direct Integration x 10 ⁻³	Galerkin x 10 ⁻³	Petrov-Galerkin x 10 ⁻³	Sub-Domain x 10 ⁻³	Least Square x 10 ⁻³	Ritz x 10 ⁻³
0.00	0	0	0	0	0	0
0.25	2.66	2.25	2.27	2.49	2.55	2.25
0.50	5.68	5.51	5.11	5.49	5.56	5.51
0.75	9.09	9.04	8.50	9.01	9.12	9.04
1.00	12.97	13.07	12.47	13.05	13.14	13.07
1.25	17.36	17.62	16.99	17.60	17.66	17.62
1.50	22.34	22.67	22.08	22.67	22.68	22.67
1.75	27.90	28.24	27.74	28.25	28.19	28.24
2.00	34.37	34.32	33.95	34.34	34.19	34.32

```

F=input('Enter load acting on body in
"N":-');
E=input('Enter the value of Youngs modulus
in "GPa":-');
E=E*(10^9); % to convert in GPa
A=(0.0005)*(exp(-x/2));% Cross-sectional
Area
x1=input('The starting position of
deformation in "m":-');
x2=input('The ending position of
deformation in "m":-');
n=input('The number of parts the body is to
be divided:-');
X=x1:(x2/n):x2; % produces array of
distance from fixed end
a=5*(10^4);
D=(R/(E*(a*(exp(-1)))));% First differential
of u wrt x
c1=(D*(E*(a*(exp(-1)))));% Constant of
integration 1
c2=-(4*c1*(exp(x1/x2))/(E*(a)*x2));%
Constant of integration 2
k1=(4*(exp(X/x2))*c1)/((E)*x2*5*(10^4));
ue=k1+c2;% Equation of deformation

```

APPROXIMATION METHODS

Approximate methods use numerical technique to obtain the solution from the governing equations. The present section discusses the issue of the finding the field variable deviation for the various positions on the continuum. The different standard numerical techniques are applied to these one dimensional continuum systems to determine results at different nodes.

Method of Weighted Residuals

The weighted residual methods are based on the assumption of an approximate solution for the governing differential equation. The assumed solution must satisfy the initial and boundary conditions of the given problem. As the assumed solution is not exact, substitution of the trial solution into the differential equation will lead to some residuals or errors. Each residual method requires the error to vanish over some selected intervals or at desired points called nodes.

The governing equation for the above considered problem is equal to residual (R) when, $u = u_{\text{approximate}}$ where $u_{\text{approximate}}$ is a polynomial function which describes the solution at the discrete points (Petrolito, 1998).

The governing equation for these methods is

$$\frac{d}{dx} \left(EA \frac{du_{\text{approximate}}}{dx} \right) = R \quad \dots(12)$$

Considering trial approximate solution,

$$u = u_{\text{approximate}} = a_0 + a_1x + a_2x^2 \quad \dots(13)$$

$$\therefore \frac{du_{\text{approximate}}}{dx} = a_1 + 2a_2x \quad \dots(14)$$

Applying first B.C. $u = 0$ at $x = 0$

$$\therefore a_0 = 0$$

Therefore the Equation (13) becomes

$$u = u_{\text{approximate}} = a_0 + a_1x + a_2x^2 \quad \dots(15)$$

This function is subsequently used for further calculation.

The weighted residual methods uses virtual work principle and the integrals error function are defined as

$$\therefore \int_0^L W_i R dx = 0, i = 1, 2, 3, \dots \quad \dots(16)$$

$$\therefore \int_0^L W_i \times \frac{d}{dx} \left(EA \frac{du}{dx} \right) dx = 0$$

The equation is integrated by parts

$$\therefore \left\{ \left(W_i AE \frac{du}{dx} \right)_{x=L} - \left(W_i AE \frac{du}{dx} \right)_{x=0} \right\} - \int_0^L EA \frac{du}{dx} \frac{dW_i}{dx} dx = 0 \quad \dots(17)$$

where W_i are the weights, which are different for each method.

The constants of trial solution a_1 and a_2 are calculated by solving Equation (12) for different weights. The deformation is calculated by substituting them in Equation (15).

$$\therefore U = U_{\text{approximate}} = a_1 x + a_2 x^2$$

In the developed program the constants are ' a_1 ' and ' a_2 ' are declared. The letter ' x ' is variable in ' u ' of trial solution assumed. The first order differentiation of ' u ' is calculated as ' du '. The weights are defined as ' w_1 ' and ' w_2 ' which will modify as per the methods considered for analysis. The differentiation of these weights is calculated as ' dw_1 ' and ' dw_2 '. The differential equation ' $f12$ ' is formulated by building the equation in terms of ' $f11$ '. Similarly the second equation ' $f22$ ' is formulated and integrated over the domain. The integration provides us with two simultaneous equations ' $c1$ ' and ' $c2$ '. These equations are solved simultaneously and their values are saved in ' a ', which is an array holding the solution. The constants of trial solution are retrieved from it and the trial solution is calculated for different values of ' X ', the distance of a point from fixed end. The common syntax applicable for the

different Method of Weighted Residuals is depicted below with the appropriate comments with each statement.

```
syms x;
A=(0.0005)*(exp(-x/2)); % Cross-sectional Area
syms a1 a2;
u=(a1*x)+(a2*(x^2)); % Trial solution
du=diff(u,x); % First differential of u wrt x
w1= %**% Weight defined by particular method
dw1=diff(w1,x); % First differential of w1 wrt x
f11=diff((w1*F),x); % Part 1 of differential eq(1)
f12=(dw1*A*E*du); % Part 2 of differential eq(1)
c1=int(f11,x,x1,x2)-int(f12,x,x1,x2); % Total differential eq(1)
w2= %**% Weight defined by particular method
dw2=diff(w2,x); % First differential of w2 wrt x
f21=diff((w2*F),x); % Part 1 of differential eq(2)
f22=(dw2*A*E*du); % Part 2 of differential eq(2)
c2=int(f21,x,x1,x2)-int(f22,x,x1,x2); % Total differential eq(2)
a=solve(c1,c2,a1,a2); % Solving eq(1) & (2) simultaneously
aa1=a.a1; % Constant 1 of Trial solution
aa2=a.a2; % Constant 2 of Trial solution
uu=(aa1*X)+(aa2*(X.^2)); % Equation of Trial solution
```

The above syntax will remain same only with the changes in the weight used corresponding to the method of selection considered for execution.

Galerkin Method

Galerkin Method is a Weighted Residual Methods which derives its weight functions from the trial solution.

Weight function is given by

$$W_i = \frac{du_{approximate}}{da_i} \quad i = 1, 2, 3, \dots \quad \dots(18)$$

Therefore

$$W_1 = \frac{du_{approximate}}{da_1} \text{ and } W_2 = \frac{du_{approximate}}{da_2}$$

Integral of the error function is $\int_{\Omega} W_i \cdot R \cdot dx = 0$ (Cockburn *et al.*, 2000).

Considering the weight functions as W_1 and W_2 and finding the constants, the trial solution is obtained.

The weights for this method can be calculated in form of

`w1_gal=diff(u_gal,a1_gal); %
Weight(1)obtained from u`

`w2_gal=diff(u_gal,a2_gal); %
Weight(2)obtained from u`

The constants are determined and field equation obtained is,

$$u_{approximate} = 8.9158 \times 10^{-6} x + 4.0871 \times 10^{-6} x^2$$

Displacements at various points are determined and represented in Table 1 and the graphically results are represented in Figure 4.

Petrov-Galerkin Method

Petrov-Galerkin Method works on similar approach which uses different pairs of weight functions such as $W_i = 1$ and x , x and x^2 , 1 and x^2 , etc.

Considering the weight functions as x^2 and x^3 for further calculation.

The weights for this method can be written in program as

`w1_petro_gal=x^2; % First Weight
considered`

`w2_petro_gal=x^3; % Second Weight
considered`

Again the same integral error function is used which is symbolized as

Integral of the error function is $\int_{\Omega} W_i \cdot R \cdot dx = 0$ (Cockburn *et al.*, 2000).

The constants are determined and field equation obtained is,

$$u = 8.9158 \times 10^{-6} x + 4.0871 \times 10^{-6} x^2$$

Displacements at various points is depicted in Table 1 and the graphically results are represented in Figure 4.

Sub-Domain Method

The domain of Figure 1 is divided in two halves and the analysis is carried out with single weight function (Mikhlin, 1963).

Considering weight as $w = x$, $\therefore \frac{dw}{dx} = 1$

The domain is 0 to 2 m which is divided into two halves from 0 m to 1 m and 1 m to 2 m. The governing Equation (17) used in this method which is multiplied by considered weight function. The equation is expressed as

$$\int_0^L (W \times R) dx = 0 \quad \dots(19)$$

For sub domain 0 to $\frac{L}{2}$ and $\frac{L}{2}$ to L Equation (19) gives two equations which when solved simultaneously provides the constants of trial solution a_1 and a_2 . Additional variable in term of $x_3 = (x_2 - x_1)/2$; is defined to indicate intermediate limit of integration.

To execute the program following changes are carried out.

$$c1 = \text{int}(f11, x, x1, x2) - \text{int}(f12, x, x1, x3); \% \text{eq}(1)$$

$$c2 = \text{int}(f21, x, x1, x2) - \text{int}(f22, x, x3, x2); \% \text{eq}(2)$$

The constants are determined and field equation obtained is,

$$u_{\text{approximate}} = 9 \times 10^{-6} x + 4.09 \times 10^{-6} x^2$$

Displacements at various points is depicted in Table 1 and the graphically results are represented in Figure 4.

Least Square Method

The Least Square Method is a weighted residual method which derives its weight functions from the Residual itself. Weight function is given by

$$W_i = \frac{dR}{da_i} \quad i = 1, 2, 3, \dots \quad \dots(20)$$

Therefore

$$W_1 = \frac{dR}{da_1} \text{ and } W_2 = \frac{dR}{da_2}$$

Weights are multiplied with Residual and integrated over the domain (Cockburn *et al.*, 2000).

The weights for this method can be calculated in form of

$$\text{res} = \text{diff}((E \cdot A \cdot du_les), x); \% \text{Residual}$$

$$w1_les = \text{diff}((\text{res}), a1_les); \% \text{Weight}(1) \text{ obtained from Residual}$$

$$w2_les = \text{diff}((\text{res}), a2_les); \% \text{Weight}(1) \text{ obtained from Residual}$$

The constants are determined and field equation obtained is,

$$u_{\text{approximate}} = 9.1946 \times 10^{-6} x + 3.9497 \times 10^{-6}$$

Displacements at various points is depicted in Table 1 and the graphically results are represented in Figure 4.

Ritz Method

In continuum mechanics, a system can be described in terms of an “energy functional”, which measures the energy of proposed configuration. It is often impossible to analyze all of the infinite configurations of system with the least amount of energy; an approximate numerical computation is essential (Mikhlin, 1963). Ritz method is applied to the above discussed static equilibrium problem. The trial functions selected above is substituted in to the functional. The f is equal to summation of strain energy and applied work potential.

Functional governing equation is

$$f = \int_0^L \frac{1}{2} \times AE \left(\frac{du}{dx} \right)^2 dx - \{U_{x=L=2} \times F\} \quad \dots(21)$$

Now differentiate Equation (21) with constants of trial function a_1 and a_2 ,

$$\frac{\partial f}{\partial a_1} = 0 \quad \dots(22)$$

and

$$\frac{\partial f}{\partial a_2} = 0 \quad \dots(23)$$

Solving simultaneous Equations (22 and 23) the constants are determined and field equation obtained is,

$$U_{approximate} = 1.534 \times 10^{-5} x + 1.111 \times 10^{-6} x^2$$

Displacements at various points is depicted in Table 1 and the graphically results are represented in Figure 4.

Program Code for Ritz Method

The program calculates the deformation using Ritz method. The approaches are similar to MWR. The variables which are required to define formulated for the trial solution are selected. The f function is calculated by evaluating ' $p1$ ', ' $p2$ ' and ' ulr '. Two equations are formed by differentiating the ' $p1$ ' function with respect to the constants of trial solution. The simultaneous equation is solved to calculate the value of ' $a1$ ' and ' $a2$ '.

```
syms x a1 a2
u=(a1*x)+(a2*(x^2));% Trial solution
du=diff(u,x);% First differential of u wrt x
p1=(0.5*E*A*(du^2));% Part 1 of Pi inside function
ulr=((a1*x2)+(a2*(x2^2)))R;% Part 2 of Pi function
p2=int(p1,x,x1,x2);% Part 1 of Pi final function
pi=p2-ulr;% Total Pi function
d1pi=diff(pi,a1);% First differential of pi wrt a1 eq(1)
d2pi=diff(pi,a2);% First differential of pi wrt a2 eq(2)
a=solve(d1pi,d2pi,a1,a2);% Solving eq(1) & eq(2) simulteneosly
```

```
a1=a.a1;% Constant 1 of Trial solution
```

```
a2=a.a2;% Constant 2 of Trial solution
```

```
uu=(a1*X)+(a2*(X.^2));% Equation of Trial solution
```

Executing the program the displacements at various points is depicted in Table 1 and the graphically results are represented in Figure 4.

Above Table 1 indicates the comparison of results between different method. Out of which we can conclude "Lest Square Method" to be more accurate on observing the above table we come to know that the result of first two field variable are more precise while there is more deviation of the result in the intermediate phase, mean while the result of "Galerkin Method" are accurate but for few field variables and again for the extreme phase we get precise result for "Lest Square Method".

The percentage error is calculated with respect to direct integration method and it is represented in Table 2.

Table 2: Percentage Error in Deformations

Methods →	Galerkin	Petrov- Galerkin	Sub- Domain	Least Square	Ritz
Distances (m) ↓					
0.25	0.15410	0.14662	0.06391	0.04135	0.1541
0.5	0.02990	0.10035	0.03345	0.02110	0.0299
0.75	0.00550	0.06491	0.00880	-0.00300	0.0055
1	-0.00770	0.03800	-0.00610	-0.01300	-0.0070
1.25	-0.01498	0.02130	-0.01380	-0.01700	-0.0140
1.5	-0.01480	0.01100	-0.01470	-0.01500	-0.0140
1.75	-0.01220	0.00500	-0.01250	-0.01000	-0.0120

CONCLUSION

The issue of selecting the numerical methods to obtain the approximate solution for the system is addressed with different numerical

approximation methods. The weighted residual methods explicitly include the errors in the field variable which is found out using the developed program. Fine variation in the field variable is possible with the developed program by using the desired steps of increments. The best convergence is found out by executing these numerical techniques to select the appropriate regions of operations. The smooth convergence close to the exact solution is essential for a best possible selection of trial solution which is possible with the developed program. Selection of trial function can be forecasted based on the results of field variable error percentage. The computer programming technique is employed to minimize computational cost and time. 🌀

REFERENCES

1. Ames W F (1992), *Numerical Methods for Partial Differential Equations*, 3rd Edition, Academic Press, New York.
2. Cockburn B, Karniadakis G E and Chi-Wang Shu (2000), "Discontinuous Galerkin Method: Theory, Computation and Applications", Springer-Verlag, Berlin.
3. Day J T and Collins G W II (1964), "On the Numerical Solution of Boundary Value Problems for Linear Ordinary Differential Equations", *Comm. A.C.M.*, Vol. 7, pp. 22-23.
4. Finlayson B A (1972), "The Method of Weighted Residuals and Variational Principles", Academic Press, New York.
5. Hamming R W (1962), "Numerical Methods for Scientists and Engineers", pp. 204-207, McGraw-Hill Book Co., Inc., New York, San Francisco, Toronto, London.
6. Hughes T J R, Angel G, Mazzei L and Larson M G (2000), "A Comparison of Discontinuous and Continuous Galerkin Method Based on Error Estimates, Conservation, Robustness and Efficiency", In *Discontinuous Galerkin Method: Theory, Computation and Application*, pp. 135-136, Springer-Verlag, Berlin.
7. Marquardt D W (1963), "An Algorithm for Least-Squares Estimation of Nonlinear Parameters", *J. Soc. Ind. Appl. Math.*, Vol. 11, No. 2, pp. 431-441.
8. Mikhlin S G (1963), "Variational Methods in Mathematical Physics", Pergamon Press, New York.
9. Otto S R and Denier J P (xxxx), "An Introduction to Programming and Numerical Methods in MATLAB", Springer-Verlag, Berlin.
10. Petrolito J (1998), "Approximate Solutions of Differential Equations Using Galerkin's Method and Weighted Residuals", Australia, pp. 14-25.
11. Pian T H H and Tong P (1969), "Basis of Finite Element Method for Solid Continua", *International Journal for Numerical Methods in Engineering*, Vol. 1, pp. 3-28.
12. Press W H, Flannery B P, Teukolsky S A and Vetterling W T (1986), "Numerical Recipes the Art of Scientific Computing", pp. 563-569, Cambridge University Press, Cambridge.