International Journal of Mechanical Engineering and Robotics Research

ISSN 2278 – 0149 www.ijmerr.com
Vol. 1, No. 2, July 2012

Research Paper

MINIMUM MASS DESIGN OF COMPOUND CANTILEVER COLUMNS WITH BUCKLING LOAD CONSTRAINT THROUGH NUMERICAL SEARCH

V Sridhar Patnaik[1], G Venkateswara Rao[2*] and A V S S K S Gupta[3]

Corresponding Author: **G Venkateswara Rao, ✉ hydrao1944@yahoo.co.in*

Minimum mass configurations, with a buckling load constraint, of the two stepped cantilever column, used in many fields of engineering, with the single or two materials (compound) in each segment of the step, are obtained. A mathematical formulation, to evaluate the buckling loads of the columns with the single or two different materials, used in the segments of the two steps, is presented. The constituent materials used in the design of the two stepped columns are aluminum, steel, copper and titanium, either independently or in combination. The minimum mass obtained for the two stepped steel cantilever column, by using the numerical search algorithm, compare very well with that evaluated by the exact and the finite element solution. The advantage of using the single the two different materials in the segments, in the minimum mass design of the compound two stepped columns, is established.

Keywords: Buckling, Two stepped columns, Different materials, Buckling load constraint, Minimum mass design

INTRODUCTION

The study of the phenomenon of buckling of the commonly used structural members, like the columns subjected to a compressive concentrated load at the free end, used in many fields of engineering, like the aerospace, automobile, mechanical, civil etc. is important, when the design is based on the stiffness criterion. It is well known that a uniform cantilever column with a constraint on the buckling load is not an optimum one with respect to the mass. The continuously varying cross-section with a convex distribution of the product of the Young's modulus and area moment of inertia, starting from zero at the free end and maximum at the fixed end has to be adapted to achieve a minimum mass design (Wang *et al.*, 2004). In the present study, the

1 Department of Mechanical Engineering, Aurora's Engineering College, Bhongir-508 116, India.
2 Department of Mechanical Engineering, Vardhaman College of Engineering, Shamshabad, Hyderabad-501 218, India.
3 Department of Mechanical Engineering, Jawaharlal Nehru Technological University, Hyderabad-500 085, India.

minimum mass design of the two stepped cantilever columns of circular cross - section, with two uniform segments of different lengths and diameters, which are much simpler production – wise, when compared to the continuously varying cross-section of the columns, is attempted. The segments of the columns are made up of either with a single material or with the two materials. The materials used in the production of these columns are aluminum, steel, copper or titanium. From now onwards, to avoid repetition, the two stepped columns with the same material in the two segments are denoted as stepped columns and the same with the two materials in the two segments are denoted as compound stepped columns.

In the present work, a similar mathematical formulation, proposed in (Jang and Bert, 1989), to study the free vibration behavior of the stepped hinged – clamped beams, is modified to obtain the buckling loads of the cantilever columns, subjected to a concentrated axial compressive load at the free end. The modified solution procedure, to predict the buckling load of the stepped and compound stepped columns, is given in this study. The minimum mass designs of the stepped and the compound stepped columns are obtained by using the numerical search algorithm, which is easier than the other optimization techniques (Thompson and Hunt, 1973), when the number of design variables are less. In the present study, the number of the design variables involved in the optimization problem of the stepped and compound stepped columns are two, namely, the length and the inertia ratios $\left(\frac{L_2}{L_1} \text{ and } \frac{I_2}{I_1}\right)$ of the segments of the column, as shown in Figures 1a and 1b. The constituent materials used in the stepped column are steel; whereas the constituent materials used in the compound stepped column are the combination of the aluminum – steel, aluminum – copper and aluminum – titanium, keeping aluminum as the free end segment. The minimum mass obtained for the stepped steel column, with a buckling load constraint (P^*_{cr}), compare very well with those available in the literature (Tadjbakhsh and Keller, 1962; and Rao and Swami, 1980). The advantage of using the two different materials in the compound stepped column, to obtain the minimum mass design, is brought out, in this paper.

In this paper, the authors have concentrated on the determination of the minimum mass configuration of stepped and stepped compound columns for the given buckling load constraint (P^*_{cr}) by solving the corresponding fourth order differential equation by taking care of the boundary conditions at the ends of the column and the continuity condition at the step. The constituent materials, if used judiciously, in the different segments of the column, find applications to arrive at the minimum mass configurations in the areas of power generation, chemical, petrochemical, nuclear, aerospace, transpor-tation, electronics, etc. (Seli *et al.*, 2010; and Date *et al.*, 1999).

One of the major problems faced in realizing the compound columns is the joining of the different materials efficiently. This is effectively achieved by the modern welding techniques developed by many researchers. The different joining techniques, especially friction welding, friction stir welding, explosive welding, cold roll welding and resistance spot welding are dealt in (Abbasi *et al.*, 2001; Aritoshi *et al.*, 1991; Cowan *et al.*, 1971; Dawes, 1977; Taban *et al.*, 2010; Seli *et al.*, 2010; Date *et al.*, 1999; Qiu *et al.*, 2010; Yilbas *et al.*, 1995; and Luo and Acoff, 2000). The advantage of these welding techniques is that there is no heat affected zone in these welding processes and the slender beams can be welded efficiently using these methods (Midling and Grong, 1994; Watanabe *et al.*, 2006; and Lee *et al.*, 2006).

In the present study, the minimum mass configurations of the two stepped single material steel, aluminum, copper and titanium cantilever columns and the two stepped compound columns, namely, aluminum – steel, aluminum – copper and aluminum – titanium columns are obtained by suitably modifying the exact procedure given in (Jang and Bert, 1989), for solving the free vibration problem of two stepped (BC)s beam made of a single material. The minimum mass obtained with a given buckling load constraint for the two stepped cantilever steel column match very well with the literature values (Tadjbakhsh and Keller, 1962; and Rao and Swami, 1980). This gives confidence in obtaining the minimum mass configurations of the compound stepped columns with the same buckling load constraint. The effect of the material used in the fixed end segment with the free end aluminum segments of the compound stepped column is clearly brought out.

FORMULATION

The differential equation governing the buckling of the columns (Thompson and Hunt, 1973), is

$$EI\frac{d^4 y}{dx^4} + P\frac{d^2 y}{dx^2} = 0 \qquad ...(1)$$

where *EI* is the flexural rigidity, P is the axial end concentrated compressive load acting along the x-axis and y represents the lateral deflection of the column.

In the present study, the formulation presented is for the two stepped single material and the compound columns having two

Figure 1a: Two Stepped Steel Column

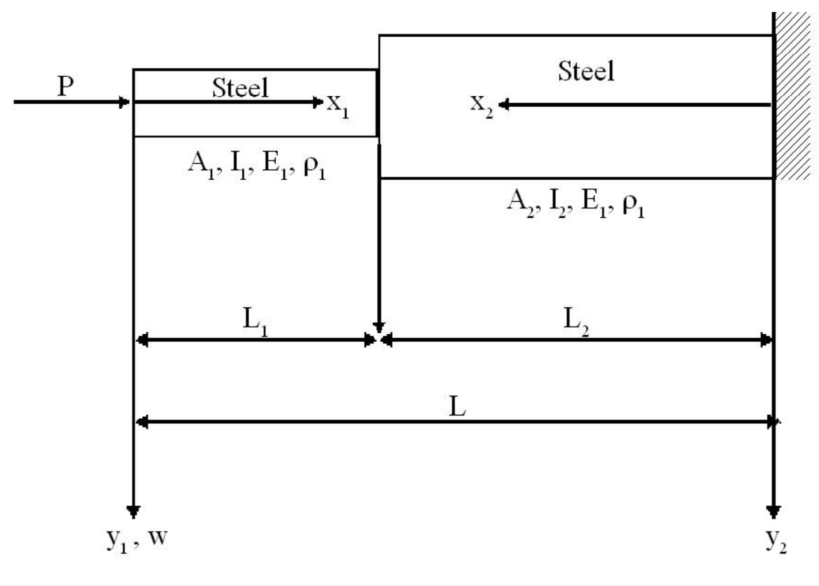
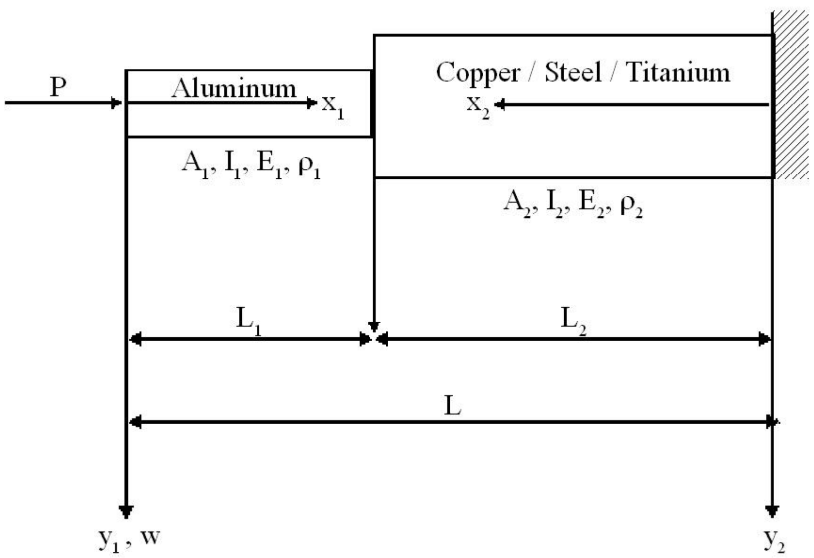


Figure 1b: Two Stepped Compound Column



segments of lengths L_1 and L_2, with free-clamped ends as shown in Figure 1a. In the case of the two materials, the segment of length L_2 and area A_2 is made of a material having higher modulus of elasticity such as steel, copper and titanium as shown in Figure 1b.

For the stepped column, as shown in Figure 1b, Equation (1) can be written, as

$$E_i I_i \frac{d^4 y_i}{dx_i^{\,4}} + k_i^{\,2} \frac{d^2 y_i}{dx_i^{\,2}} = 0 \quad ...(2)$$

where $k_i^2 = P/E_i I_i$ and $i = 1, 2$ represents the first and the second segments of the two stepped column.

The general solution of the differential equation given by Equation (2) is obtained, as

$$y_1 = C_1 \sin k_1 x_1 + C_2 \cos k_1 x_1 + C_3 x_1 + C_4$$

$$y_2 = C_5 \sin k_2 x_2 + C_6 \cos k_2 x_2 + C_7 x_2 + C_8 \quad ...(3)$$

The boundary conditions considered for the cantilever conditions are

$$E_1 \quad I_1 \frac{d^2 y_1}{dx_1^{\,2}} = 0 \text{ and } \frac{d^3 y_1}{dx_1^3} + k_1^2 \frac{dy_1}{dx_1} = 0$$

$$\text{at } x_1 = L_1 \quad y_2 = \frac{dy_2}{dx_2} = 0 \text{ at } x_2 = 0 \quad ...(4)$$

The conditions at the step are written by using the conditions on the lateral deflection, slope, moment and shear force are

$$y_1(L_1) = y_2(L_2), \quad \frac{dy_1}{dx_1}(L_1) = -\frac{dy_2}{dx_2}(L_2),$$

$$E_1 \quad I_1 \frac{d^2 y_1}{dx_1^2}(L_1) = E_2 \quad I_2 \frac{d^2 y^2}{dx_2^2}(L_2)$$

$$\frac{d}{dx_1}\left(E_1 I_1 \frac{d^2 y_1}{dx_1^2} \right)(L_1)$$

$$= -\frac{d}{dx_2}\left(E_2 I_2 \frac{d^2 y_2}{dx_2^2} \right) \ (L_2) \quad ...(5)$$

TWO STEPPED CANTILEVER COLUMN

For the stepped free-clamped column, the relations between the constants of integration, obtained from Equations (3) and (4) are $C_2 = C_3 = 0$, $C_6 = -C_8$ and $C_7 == -C_5 k_2$.

Using the quantities $k = k_2/k_1$ and $I = I_2/I_1$, from the Equations (3) to (5), the following matrix equation can be obtained, as

$$\begin{bmatrix} \sin k_1L_1 & 1 & k_2L_2 - \sin k_2L_2 & 1-\cos k_2L_2 \\ (\cos k_1L_1) & 0 & k(\cos k_2L_2 - 1) & -k\sin k_2L_2 \\ \sin k_1L_1 & 0 & -\sin k_2L_2 & -\cos k_2L_2 \\ \cos k_1L_1 & 0 & k\cos k_2L_2 & -k\sin k_2L_2 \end{bmatrix}$$

$$\begin{bmatrix} C_1 \\ C_4 \\ C_5 \\ C_6 \end{bmatrix} = 0 \quad \text{... (6)}$$

SOLUTION PROCEDURE

1. The nontrivial solution of the matrix Equation (6), i.e., the design diameterD2, is determined iteratively by using the software Matlab7.1 for the known values of the buckling load (P^*_{cr}),area moment of inertia (I_1) of segment1 and length ratio (L_2/L), varying from 0.1 to 1 in steps of 0.1.
2. Mass of the beam is calculated for the obtained values of D2 by using Equation (7).
3. As can be seen, the values in Table 3, reference values show good agreement with the exact solutions.(less than 4% error).

$$M = \rho_1 A_1 L_1 + \rho_2 A_2 L_2 \quad \text{...(7)}$$

NUMERICAL RESULTS AND DISCUSSION

Figure 1a shows a free-clamped stepped steel column with two segments and the compound stepped column with two segments is shown Figure 1b.For the compound stepped column, the free end segment of length L_1, cross-sectional area A_1 and the area moment of inertia I_1 is made of aluminum and for the clamped end segment of length L_2, cross-sectional area A_2 and the area moment of inertia I_2 is made of copper, steel and titanium. The material properties, namely, the Young's modulus and density of aluminum, copper, steel and titanium, considered in this work are presented in Table 1. The buckling loads of the compound columns subjected to an end concentrated compressive loads are evaluated by using the present formulation. The following dimensions of the column are taken

Table 1: Mechanical Properties of Aluminum, Steel, Copper and Titanium

Material	Young's Modulus GPa	Density kg/m^3
Aluminum	69	2780
Steel	204	7750
Copper	110	8940
Titanium	115	4429

in the present study: The length of the column is 0.254 m (100 in) and the diameter of the aluminum segment is 30.48 mm (1.2 in).

For all the material configurations, of the stepped free-clamped or compound stepped column the buckling load constraint (P^*_{cr}) is taken as 453.59 kgf (1000 lb).

The mass of the stepped column, made of steel, is shown in Table 2. The minimum mass obtained for this beam 14.9787Kg for the length and inertia ratios of 0.44 and 1.2040 respectively. As shown in Table 3 ,the value of the minimum mass for the configuration of the eight stepped steel beam, using the symmetry, is 14.5771 kg [3], obtained by employing the finite element based optimality criterion approach and the exact value of 14.4185Kg[2], evaluated by Tadjbakhsh and Keller[2]through a continuum analysis. The minimum mass obtained in the present study differs from the exact value by 3.885% and from the Rao and Swamy (1980) by 2.75%. This difference is due to the fact that the two stepped configuration of the beam introduces a strong geometric constraint, when compared to the exact solution of Tadjbakhsh and Keller (1962), where the distribution of area moment of inertia is a continuous convex curve. This phenomenon is also seen in the solution obtained by the finite element method coupled

Table 2: Mass of Two Stepped Steel Cantilever Column

Length Ratio L_2/L	Diameter D_2 m	Inertia Ratio I_2/I_1	Mass kg
0.10	0.10431	1.3719	29.7519
0.20	0.04091	3.2455	16.6666
0.30	0.03577	1.8959	15.9883
0.40	0.03422	1.5887	15.8604
0.42	0.03404	1.5550	15.8540
0.44	0.03193	1.2040	14.9787
0.46	0.03374	1.5015	15.8531
0.48	0.03362	1.4801	15.8575
0.50	0.03351	1.4616	15.8649
0.60	0.03315	1.3987	15.9384
0.70	0.03295	1.3665	16.0629
0.80	0.03286	1.3513	16.2309
0.90	0.03283	1.3459	16.4341
1.00	0.03282	1.0000	16.6594

Table 3: Summary of Minimum Mass Designs of Two Stepped Cantilever Columns with Single Material Subjected to Buckling Load Constraint of 453.5927 kg (1000 lb)

Parameter	Length Ratio L_2/L	Inertia Ratio I_2/I_1	Minimum Mass kg	Minimum Volume m^3
Steel	0.44	1.2040	14.9787 (14.577)* (14.418)$ (15.829)#	0.00193

Note: $ Tadjbakhsh and Keller [2]; * Rao G V and Swami R [3]; # Rao G V *et al.* [22]

Table 4: Mass of Two Stepped Compound Aluminum – Steel Cantilever Column

Length Ratio L_2/L	Diameter D_2 m	Inertia Ratio I_2/I_1	Mass kg
0.10	**		
0.20	**		
0.30	**		
0.40	**		
0.50	0.07033	28.35250	40.7558
0.60	0.03749	2.28992	15.0528
0.70	0.03422	1.58908	14.1832
0.72	0.03391	1.53244	14.2103
0.74	0.03366	1.48789	14.2727
0.76	0.03346	1.45277	14.3645
0.78	0.03330	1.42505	14.4808
0.80	0.03317	1.40331	14.6181
0.90	0.03287	1.35173	15.5329
1.00	0.03283	1.00000	16.6594

Note: ** no solution available.

with the optimality criterion approach, where the beam is idealized into eight elements (segments) in the half of the beam due to symmetry of the beam configuration. This shows that if the steps are more, the induced geometric constraint becomes milder and the value obtained for the optimum mass of the multi-stepped beam will be closer to the exact value, when compared to the present two stepped configuration

As has been already mentioned and shown in Figure 1b, the main motivation of the present work is to obtain the minimum mass configuration of the stepped symmetric configuration of the column through a parametric study by changing the parameters $\frac{L_2}{L}$ and $\frac{I_2}{I_1}$ representing the length and inertia ratios. Table 4 shows the mass of an aluminum-steel compound stepped column. The optimum value of the mass of 14.183 kg is obtained for a beam of length and inertia ratios of 0.7 and 1.5891 respectively. Please note that it is not the aim of the present study to show that a stepped compound steel beam will replace the stepped steel column, but to emphasize that the stepped compound column gives a lesser optimum mass compared to the steel beam, for which the exact optimum design is available.

Table 5 gives the optimum value of an aluminum-copper stepped compound column. The minimum mass of 21.457 kgf is obtained for the length and inertia ratios of 0.7 and 2.9398 respectively. However, this is not a usable design though feasible. These two results give a clue on the minimum mass

Table 5: Mass of Compound Stepped Aluminum – Copper Cantilever Column

Length Ratio L_2/L	Diameter D_2 m	Inertia Ratio I_2/I_1	Mass kg
0.10	**		
0.20	**		
0.30	**		
0.40	**		
0.50	0.08203	52.45200	62.7023
0.60	0.04373	4.23627	22.5371
0.70	0.03991	2.93983	21.4573
0.72	0.03955	2.83499	21.5575
0.74	0.03926	2.75259	21.7135
0.76	0.03903	2.68757	21.9155
0.78	0.03884	2.63637	22.1565
0.80	0.03869	2.59619	22.4305
0.90	0.03833	2.50073	24.1586
1.00	0.03828	1.00000	26.2199

Note: ** No solution available.

design of a stepped compound column, with the clamped end segment made of different materials, is based on the given inertia ratio and for a particular length ratio. This concept is further justified by considering an aluminum-titanium stepped compound column, for which the minimum mass obtained is11.177 kg at the length ratio of 0.72 (Table 6).The inertia ratio evaluated for this stepped compound column is 2.7326 and is lower than those of the earlier considered stepped compound columns.

The comparisons of the stepped compound columns are shown in the Table 7.

Table 6: Mass of Compound Stepped Aluminum – Titanium Cantilever Column

Length Ratio L_2/L	Diameter D_2 m	Inertia Ratio I_2/I_1	Mass kg
0.10	**		
0.20	**		
0.30	**		
0.40	**		
0.50	0.08128	50.55590	31.6943
0.60	0.04333	4.08317	11.9625
0.70	0.03955	2.83361	11.1800
0.72	0.03919	2.73256	11.1768
0.74	0.03890	2.6531	11.2006
0.76	0.03873	2.60616	11.2771
0.78	0.03854	2.55647	11.3426
0.80	0.03834	2.50232	11.3927
0.90	0.03798	2.41031	11.9715
1.00	0.03793	1.00000	12.7121

Note: ** No solution available.

Table 7: Summary of Minimum Mass Designs of Compound Stepped Cantilever Columns

Parameter	Material		
	Aluminum – Steel	Aluminum – Copper	Aluminum – Titanium
Length Ratio L_2/L	0.70	0.70	0.72
Inertia Ratio I_2/I_1	1.5891	2.9398	2.7326
Minimum Mass kg	14.183	21.457	11.177

CONCLUSION

The concept of a stepped compound column is proposed in this paper to obtain a near optimum (minimum mass) configuration of the stepped compound column. Three materials, aluminum, copper, steel and Titanium are mainly considered, in this paper. The optimum configuration is obtained through a parametric study and is adequate for the engineering applications considering the inevitable non-optimum mass. The advantage of this concept is that the manufacturing process involved in obtaining a desired continuous change of the area moment of inertia of the beam is avoided. The same analysis on a stepped steel column shows the advantage of using a stepped compound column. From the results obtained from the present work on the stepped compound columns, though restricted in number, it can be concluded that the minimum mass design seems to be strong function of the inertia ratio. This idea is further strengthened by considering the aluminum-titanium stepped compound column. Based on this study, the following categorical statement can be made: The lesser the inertia ratio the lesser the minimum mass of the stepped compound beam. It is to be noted here that for each inertia ratio the minimum mass occurs at a particular value of the length ratio.

ACKNOWLEDGMENT

The authors gratefully acknowledge the managements of the respective Institutes for their constant encouragement during the course of this work.

REFERENCES

1. Abbasi M, Taheri K A and Salehi M T (2001), “Growth Rate of Intermetallic Compounds in Al/Cu Bimetal Produced by Cold Roll Welding Process”, *J. Alloys Compd.*, Vol. 319, pp. 233-241.

2. Aritoshi M, Okita K, Enjo T, Ikeuchi K and Matsuda F (1991), “Friction Welding of Oxygen Free Copper to Pure Aluminum”, *Q. J. Jpn. Welding Soc.,* Vol. 9, pp. 467-474.

3. Cowan G R, Bergmann O R and Holtzman A H (1971), “Mechanism of Bond Zone Wave Formation in Explosive-Clad Metals”, *Metall. Trans.*, Vol. 2, pp. 3145-3155.

4. Date H, Kobayakawa S and Naka (1999), “Microstructure and Bonding Strength of Impact-Welded Aluminum–Stainless Steel Joints”, *Journal of Materials Processing Technology*, Vol. 85, pp. 166-170.

5. Dawes C J (1977), “Micro-Friction Welding Aluminum Studs to Mild Steel Plates”, *Metal Construction*, Vol. 9, pp. 196-197.

6. Jang S K and Bert C W (1989), “Free Vibration of Stepped Beams: Exact and Numerical Solutions”, *Journal of Sound and Vibration*, Vol. M(2), pp. 342-346.

7. Lee W, Schmuecker M, Mercardo U A, Biallas G and Jung S (2006), “Interfacial Reaction in Steel-Aluminum Joints Made by Friction Stir Welding”, *Scripta Materialia*, Vol. 55, pp. 355-358.

8. Luo J G and Acoff V L (2000), “Interfacial Reactions of Titanium and Aluminum

During Diffusion Welding", *Welding Journal*, Vol. 79, pp. 239s-243s.

9. Midling O T and Grong Ø (1994), "A Process Model for Friction Welding of Al–Mg–Si Alloys and Al–SiC Metal Matrix Composites-I, HAZ Temperature and Strain Rate Distribution", *Acta Metall. Mater. Trans.*, Vol. 42, pp. 1595-1609.

10. Qiu R, Shi H, Zhang K, Tu Y, Iwamoto C and Satonaka S (2010), "Interfacial Characterization of Joint Between Mild Steel and Aluminum Alloy Welded by Resistance Spot Welding", *Materials Characterization*, Vol. 61, pp. 684-688.

11. Rao G V and Swami R N (1980), "Optimum Design of Cantilever Columns in the Post – Buckling Region with Constraints on Axial Load an Optimality Criterion Approach", *Computers and Structures*, Vol. 12, pp. 843-848.

12. Seli H, Izani H, Ismail M, Rachman E and Ahmad Z A (2010), "Mechanical Evaluation and Thermal Modeling of Friction Welding of Mild Steel and Aluminum", *Journal of Materials Processing Technology*, Vol. 210, pp. 1209-1216.

13. Taban E, Gould J E and Lippold J C (2010), "Dissimilar Friction Welding of 6061-T6 Aluminum and AISI 1018 Steel: Properties and Microstructural Characterization", *Materials & Design*, Vol. 31, pp. 2305-2311.

14. Tadjbakhsh I and Keller J B (1962), "Strongest Columns and Iso-Perimetric Inequalities for Eigenvalues", *Transactions of the ASME*, Vol. 84, pp. 159-164.

15. Wang A, Wang C M and Reddy J N (2004), *Exact Solutions for Buckling of Structural Members,* CRC Press, Boca Raton.

16. Watanabe T, Takayama H and Yanagisawa A (2006), "Joining of Aluminum Alloy to Steel by Friction Stir Welding", *Journal of Materials Processing Technology*, Vol. 178, pp. 342-349.

17. Yilbas B S, Sahin A Z, Kahraman N and Al-Garni A Z (1995), "Friction Welding of Steel-Al and Al-Cu Materials", *J. Mater. Process. Technol.*, Vol. 49, pp. 431- 443.

Books

18. Singiresu S Rao (2003), *Engineering Optimization, 3-e,* John Willey & Sons Inc. and New Age International (P.) Ltd., New Delhi, India.

19. Thompson J M T and Hunt G W(1973), *A General Theory of Elastic Stability,* Wiley, London.

20. Timoshenko S and Gere J M (1961), *Theory of Elastic Stability,* McGraw-Hill, New York.

Journal Article in Press

21. Rao G V, Sridhar Patnaik V and Gupta A V S S K S (2012), "Near Optimum Configuration of Stepped Hinged Beams With Buckling Load Constraint", *Journal of Structural Engineering*, CSIR, India.