Robust Control Law Using H-infinity for Wheeled Inverted Pendulum Systems

Pham Tam Thanh
School of Electrical Engineering, Vietnam Maritime University
Haiphong, Vietnam

Dao Phuong Nam, Vu Van Tu, Tran Quang Huy and Nguyen Van Huong
School of Electrical Engineering, Hanoi University of Science and Technology, Hanoi, Vietnam
Email: nam.daophuong@hust.edu.vn

Abstract—Many authors have utilized Lyapunov’s direct technique to design robust adaptive controllers. In recent years, the work of applying H infinity theory to control wheeled inverted pendulums is a topic of much concern due to its underactuated, external-disturbance-prone and nonlinear model. In this paper, we propose a new control method by applying the H-infinity and Backstepping technique based on Lyapunov’s direct technique to stabilize tracking error for converging to arbitrary ball of origin. Under disturbances belonging L₂-space, the simulation results of WIP demonstrate the effectiveness of the proposed controller.

Index Terms—H∞ control, Backstepping design, Wheeled Inverted Pendulum, Linear Matrix Inequalities

I. INTRODUCTION

The wheeled inverted pendulum described in [2] includes a pair of identical wheels, a chassis, wheel actuators, an inverted pendulum and a motion control unit, within which pair of wheels and the inverted pendulum are supported by chassis. The wheel actuators rotate the wheels with respect to the chassis, and they are controlled by the motion control unit to move the vehicle and to stabilize the inverted pendulum.

The dynamic model and several control methods are presented in [3]. In this paper, we use a dynamic model of WIP built via the Newton Euler method [3]. This model is separated into two subsystems: the subsystems describe the rotation of Cart and straight motion of WIP, which is similar to Cart–Pole model. Previous authors [4] [5] approached the control of inverted pendulum based on energy function, but the disadvantage of their approach is that disturbances impacting considered system is ignored and pendulum always fluctuate around origin. Olfati-Saber [6] proposed coordinate transformation to change the Cart–Pole model to strict forward system to apply nest saturation method. This transformation will not be effective when some parameters are uncertainties or disturbances appear. Therefore, Do [2] used a similar transformation as that in [6] and combined with the nest saturation method, disturbance observer to steady error to converge to origin asymptotically with assumption of zero straight accelerator of Cart. Researchers in [7] applied properties of nonholonomic system and backstepping technique, but their technique’s drawback is in the assumption of satisfying state constraints. The instantaneous switching of control input is proposed in [8]; nevertheless, the position of Cart therein was not stable at the desired point. Separation technique for WMRs has been mentioned in [10], and authors absolutely used Lyapunov’s technique to find the control scheme.

In this paper, we apply the H∞ method in [1] for rotation motion and an additional controller has been proposed for the straight motion subsystem. The first controller has been proposed based on Lyapunov’s direct technique guarantee that tilt angle and angular velocity error converge to neighborhood of origin. In this region, we give some additional components for the second controller using H∞ theory to ensure that all state variables lie in a small arbitrary ball of origin. Finally, to avoid an instantaneous change in control torque, we propose virtual control input that is designed using the backstepping technique [9].

II. DYNAMIC MODEL

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance between two wheels</td>
<td>D m</td>
</tr>
<tr>
<td>Radius of wheel</td>
<td>R m</td>
</tr>
<tr>
<td>Moment of inertia of the wheel</td>
<td>Jw kg.m²</td>
</tr>
<tr>
<td>Moment of inertia of the chassis and pendulum about z-axis</td>
<td>Jp kg.m²</td>
</tr>
<tr>
<td>Moment of inertia of the chassis about the y-axis</td>
<td>Ju kg.m²</td>
</tr>
<tr>
<td>Moment of inertia of heading angle pendulum about z-axis</td>
<td>Jθ kg.m²</td>
</tr>
<tr>
<td>Mass of pendulum</td>
<td>m kg</td>
</tr>
<tr>
<td>Mass of chassis</td>
<td>M kg</td>
</tr>
<tr>
<td>Mass of wheel</td>
<td>Mw kg</td>
</tr>
<tr>
<td>Gravitational acceleration</td>
<td>g m/s²</td>
</tr>
<tr>
<td>Distance between the central point of the pendulum and chassis</td>
<td>l m</td>
</tr>
</tbody>
</table>

TABLE II. VARIABLE TYPE OF WIP

| Heading angle of pendulum | \( \theta \) rad |
| Tilt angle of pendulum | \( \phi \) rad |
| Torque control in left and right wheel | \( \tau_L, \tau_R \) N.m |
| Position of chassis | \( x, m \) |
| Disturbances impacting on two wheels | \( d_x, d_y \) N |

By applying Newton – Euler Approach in [1], the dynamic model of WIP is:

\[
\dot{\theta} = \frac{D}{2R J_\theta}(\tau_L - \tau_R) + \frac{D}{2J_\theta}(d_L - d_R) \quad (1)
\]

\[
ml \cos(\phi) \ddot{\theta} + \left[ M + m + 2 \left( \frac{J}{R^2} + M_m \right) \right] \ddot{x} = ml \phi^2 \sin(\phi) + \frac{\tau_L + \tau_R}{R} + d_L + d_R
\]

\[
(ml^2 + J_m) \ddot{\phi} + ml \cos(\phi) \ddot{x} = mgl \sin(\phi) \quad (3)
\]

where:

\[
J_\theta = J_\rho + D^2 \left( M_m + \frac{J_m}{R^2} \right) \quad (4)
\]

**Assumption 1:** The external disturbances impacting two wheels belong to the \( L_\infty \)-space:

\[
|d_x| < d_{x_{\text{max}}}; \quad |d_y| < d_{y_{\text{max}}} \quad (5)
\]

\[
\left[ M + m + 2 \left( \frac{J}{R^2} + M_m \right) \right] (ml^2 + J_m) - m\dot{\theta}^2 > 0 \quad (6)
\]

**Control objective:** The heading angle, position and their derivatives track their desired value and tilt angle converge to zero.

\[
\theta, \dot{\theta}, x, \dot{x} \text{ and } \dot{x}_d \text{ are desired heading angle, heading angular velocity, position and velocity, respectively. The proposed controller needs to satisfy (7) and (8).}
\]

\[
|\theta - \theta_d| \to 0, \quad |\phi - \phi_0| \to 0, \quad |x - x_d| \to 0 \quad (7)
\]

\[
|\dot{\theta} - \dot{\theta}_d| \to 0, \quad |\phi - \phi_0| \to 0, \quad |\dot{x} - \dot{x}_d| \to 0 \quad (8)
\]

if \( \dot{d}_x = 0, \quad \dot{d}_y = 0 \)

Moreover, uniform bounded in tracking error if

\[
|d_x| < d_{x_{\text{max}}}, \quad |d_y| < d_{y_{\text{max}}} \quad (9)
\]

**III. PROPOSED METHOD**

The above equations are separated into two subsystems: \( \theta \) system (12) and \( x, \phi \) system (13,14).

A. **Control Design for \( \theta \) System**

The control input in (12) and the error of heading angle are set in (15)

\[
\tau_i = u_i + \frac{2J_\theta}{D} \ddot{\theta}_{ref} \quad (15)
\]

The subsystem (12) is written as follows:

\[
\dot{\theta} = \frac{D}{2J_\theta} u_i + \frac{D}{2J_\theta} d_\theta \quad (16)
\]

Considering subsystem (16) as follows:

\[
\dot{u} = A \omega + B u_i + B_2 d_\theta \quad (17)
\]

\[
\nu = [\dot{\theta}_i, \dot{\theta}_d]^T, A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B_1 = B_2 = \begin{bmatrix} 0 \\ D \end{bmatrix}, H > 0
\]

**Theorem 1:** By selecting positive scalar \( \gamma \), there exists a matrix \( P \) satisfying inequality (18) – [1]. The control input (19) ensures (20) to guarantee tracking error to converge to origin and to obtain its robust stability.

\[
A^T P + P A + P \left( \frac{1}{\gamma^2} B_2 B_2^T - B_1 B_1^T \right) P + H^T H \leq 0 \quad (18)
\]

\[
u_i = -B_1^T P u \quad (19)
\]

\[
\int_0^T \left( \|y\|^2 + \|u\|^2 \right) dt \leq \gamma^2 \int_0^T \|u_i\|^2 dt \quad (20)
\]

**Remark 1:** Let \( P_i = T^i \), \( \gamma \) is appropriately selected. The existence of solution in (28) satisfies \( H_{\gamma} \) [1]. The system will be applied input (19) for stabilization.

Multiplying \( P_i^T \) in left and right side of (18)

\[
P_i^T A^T + A P_i^{-1} + \frac{1}{\gamma^2} B_2 B_2^T - B_1 B_1^T \right) P_i + H^T H \leq 0 \quad (27)
\]

Schur’s complement is applying in (27) to be equivalent to (28)

\[
\begin{bmatrix} (A T)^T + AT & + \frac{1}{\gamma^2} B_2 B_2^T - B_1 B_1^T \\ HT & -I \end{bmatrix} \leq 0 \quad (28)
\]

B. **Control design \( x, \phi \) system**

Subsystem (13) and (14) is equivalent to (30) and (31) by using local feedback linearization (29):
\[
\tau_z = -ml\dot{\theta}_z \sin(\theta) + \frac{m^2 \dot{g} l^2}{m l^2 + J_u} \cos(\theta) \sin(\phi) + \\
\left( M + m + 2 \left( \frac{J_u}{R^2} + M_u \right) \right) \frac{m^2 \dot{\theta}_z \cos^2(\phi)}{m l^2 + J_u}(u + \dot{x}_d)
\]  
(29)

\[
\dot{x}_v = u + \Delta
\]  
(31)

where:
\[
a = \frac{m l^2}{m l^2 + J_u}
\]  
(32)

\[
\Delta = \frac{d}{M + m + 2 \left( \frac{J_u}{R^2} + M_u \right)} \frac{m l^2 \dot{\theta}_z \cos^2(\phi)}{m l^2 + J_u}
\]  
(33)

The desired position and velocity of WIP are:
\[
x_v = x - x_d; \quad \dot{x}_v = \dot{x} - \dot{x}_d
\]

**Remark 2:** It is difficult to directly design a \( H_v \) controller for subsystems (30) (31) because of underactuated property and interconnect control input in both two equations. We propose the Lyapunov direct method for \( \phi, \dot{\phi} \) to lead to given attractor where the linearization model can be applied. The proposed \( H_v \) controller guarantees that \( x, \dot{x} \) track simultaneously to the desired value and that \( \phi, \dot{\phi} \) are bounded by neighborhood origin. Because the proposed method includes two sub-controllers, the switching must be employed in control system.

**Assumption 2:** External disturbances \( \Delta \), desired accelerometer \( \ddot{x}_d \) and parameters satisfy
\[
|\Delta| < \Delta_{\text{max}}, |\ddot{x}_d| < \ddot{x}_{d_{\text{max}}} \quad (33)
\]

\[
M + m + 2 \left( \frac{J_u}{R^2} + M_u \right) (m l^2 + J_u) - m l^2 > 0 \quad (34)
\]

**Theorem 2:** Considering subsystem (30) under (33) and (34). By applying control torque (35) found by Lyapunov’s direct method, two variables \( \phi, \dot{\phi} \) converge to the arbitrary attractor \( \Omega = \{ \Omega \in \mathbb{R}^2 \mid |\phi| \leq \varepsilon_1, |\dot{\phi}| \leq \varepsilon_2 \} \) by adjusting controller coefficients \( k_1, k_2 \) largely.

\[
u_u(\phi, \dot{\phi}) = k_1 \dot{\phi} + k_2 \phi + g \sin \phi \cos \phi
\]  
(35)

**Remark 3:** From Remark 2 and controller (35), the applied switching may cause instantaneous change of control input, leading to unreality. Therefore, a virtual input in (36) is proposed to reduce this problem significantly. The new linear state space model with virtual input is considered as:
\[
\dot{\eta} = F \eta + G \nu_u + K (\Delta - \ddot{x}_d)
\]

**Theorem 3:** The proposed virtual control input (36) will be designed and based on backstepping following (37) to satisfy Remark 3
\[
z = -\kappa (u_1 - \bar{u}) - \frac{\partial V_2}{\partial \eta} G + \frac{\partial \eta}{\partial \eta} (F \eta + G \nu_u)
\]

**Remark 4:** The proposed control input (36, 37) is the next step of work in [10].

**IV. SIMULATION RESULTS**

Firstly, we obtain the results using controller in [10] as described in Figs. 2 and 3. We continue to implement the proposed controller (36, 37) based on the scenario simulation as follows: The heading angle and tilt angle deviated from their balanced position and position of chassis is arbitrary. The objective of work is to guarantee that \( x, \dot{x}, \theta, \dot{\theta} \) track given desired strategies \( x_d, \dot{x}_d, \theta_d, \dot{\theta}_d \) with unbalanced initial state, wherein the WIP system can move to given place via state trajectory. The good performance of simulation results (Fig. 4 to Fig. 7) demonstrates the ability of the proposed control law, which executes the requirement. The parameter values and initial state in [2] are used here to simulate this system.

**TABLE III. PARAMETER VALUES IN SIMULATION**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance between two wheels</td>
<td>( D = 0.15 \text{m} )</td>
</tr>
<tr>
<td>Radius of wheel</td>
<td>( R = 0.25 \text{m} )</td>
</tr>
<tr>
<td>Moment of inertia of the wheel about the ( y )-axis</td>
<td>( J_y = 1.5 \text{kg.m}^2 )</td>
</tr>
<tr>
<td>Moment of inertia of the chassis and pendulum about the ( z )-axis</td>
<td>( J_z = 2.5 \text{kg.m}^2 )</td>
</tr>
<tr>
<td>Moment of inertia of the chassis about the ( y )-axis</td>
<td>( J_y = 1.5 \text{kg.m}^2 )</td>
</tr>
<tr>
<td>Moment of inertia of heading angle pendulum about the ( z )-axis</td>
<td>( m = 1.5 \text{kg} )</td>
</tr>
<tr>
<td>Mass of pendulum</td>
<td>( M = 5 \text{kg} )</td>
</tr>
<tr>
<td>Mass of chassis</td>
<td>( M_s = 1 \text{kg} )</td>
</tr>
<tr>
<td>Mass of wheel</td>
<td>( g = 9.8 \text{m/s}^2 )</td>
</tr>
<tr>
<td>Gravitational acceleration</td>
<td>( l = 1.2 \text{m} )</td>
</tr>
</tbody>
</table>
TABLE IV. INITIAL STATE IN SIMULATION

<table>
<thead>
<tr>
<th>Initial state variable</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta(0)$</td>
<td>0.5 rad</td>
</tr>
<tr>
<td>$\dot{\theta}(0)$</td>
<td>0 rad/s</td>
</tr>
<tr>
<td>$x(0)$</td>
<td>-1.5 m</td>
</tr>
<tr>
<td>$\dot{x}(0)$</td>
<td>0 m/s</td>
</tr>
<tr>
<td>$\phi(0)$</td>
<td>0.8 rad</td>
</tr>
<tr>
<td>$\dot{\phi}(0)$</td>
<td>0.2 rad/s</td>
</tr>
</tbody>
</table>

TABLE V. PARAMETER OF CONTROLLERS

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
<td>1.5</td>
</tr>
<tr>
<td>$H$</td>
<td>diag([100,1])</td>
</tr>
<tr>
<td>$a_1$</td>
<td>[884.43 477.63]</td>
</tr>
<tr>
<td>$\beta$</td>
<td>100</td>
</tr>
<tr>
<td>$\varepsilon_1$</td>
<td>0.3</td>
</tr>
<tr>
<td>$\varepsilon_2$</td>
<td>1</td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>2.78</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>0.25</td>
</tr>
<tr>
<td>$k_1$</td>
<td>48.41</td>
</tr>
<tr>
<td>$k_2$</td>
<td>39.27</td>
</tr>
<tr>
<td>$C$</td>
<td>diag([0.1,1.2,0.1])</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>100</td>
</tr>
<tr>
<td>$a_3$</td>
<td>[98.49,31.99,3.44,10.65]</td>
</tr>
</tbody>
</table>

This system is able to self-balance and straight velocity, heading angle track reference signal under disturbances. Control forces is continuous. H-infinity control is also able to track zero of tilt angle better than robust control.

V. CONCLUSION

The proposed approach applied in this paper is that of separating dynamic model into two subsystems, including rotation and straight to design two controller. Heading angle tracks its reference when controller is designed by $H_\infty$. Tilt angle and position reach their balanced point by combining Lyapunov’s direct method controller and $H_\infty$ controller in order to avoid instantaneous change of controllers, the virtual input is applied. The simulation results in previous chapter verify performance of the proposed method.

ACKNOWLEDGMENTS

The authors gratefully acknowledge the constructive and insightful comments of reviewers for further improve the quality of this paper. This work is supported in part by the Hanoi University of Science and Technology (T2017-PC-105)

REFERENCES


PhD. Phuong Nam Dao obtained a Doctorate degree on January, 2013, at Hanoi University of Science and Technology (Vietnam). Currently, he holds the position as lecturer at Hanoi University of Science and Technology, Vietnam. His research interests include control of robotic systems and robust/adaptive optimal control. He is the author/co-author of more than 70 papers (Journals, Conferences, etc.)

Nguyen Van Huong: Place of birth: Hai Duong, Viet Nam. Date of birth: 21/02/1995. Nguyen Van Huong received a Bachelor’s degree in Electronic Engineering in 2018 from the Hanoi university of science and technology, Vietnam. Currently, he works in research fields such as robotics, electric power systems and control systems.

Pham Tam Thanh received a Bachelor’s degree in Industrial Automation from Hanoi University of Science and Technology, Hanoi, Vietnam (HUST) in 2003, and an M.S. degree in Automation and Control Engineering from Vietnam University Maritime (VU), Vietnam, in 2008, and a Ph.D degree in automation and control engineering from HUST in 2015. He is currently with Department of Automation and Control Engineering, School of Electrical and Electronics Engineering, Vietnam Maritime University, as Full Lecture. His research interests include stability and controller design, control renewable systems, vector control AC machines and their applications, vector control of electrical drives and nonlinear control.