Time-to-go Polynomial Guidance with Impact Angle Constraint for Missiles of Time-Varying Velocity

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Abstract—In this paper, a Time-to-go Polynomial Guidance (TPG) law satisfying impact angle constraints for missiles of time-varying velocity against stationary target is investigated. The guidance law assumes the magnitude of guidance commands as a polynomial function of time-to-go with two unknown coefficients. The coefficients are determined to satisfy the two terminal constraints, which are the terminal impact angle and zero miss distance. The derivation of the guidance law is constructed for exponentially decreasing velocity. The main contribution of this study is expansion of the concept of TPG for constant velocity missiles to missiles of time-varying velocity. The proposed guidance law is identical to typical TPG when the velocity of missile is constant. To verify the performance of the proposed guidance law, numerical simulation is performed. The simulation shows that the proposed guidance law is a sub-optimal solution of minimizing timeto-go weighted energy cost function.

Index Terms—time-to-go polynomial guidance, velocity varying missiles, impact angle control

I. INTRODUCTION

Over the past several decades, many advanced guidance laws have been proposed to achieve various objectives, such as energy consumption minimization, the terminal impact angle and/or impact time control, the observability enhancement and improving guidance performance. Among the objectives, the impact angle is one of the most important constraint for homing missiles to ensure maximum warhead effectiveness and high kill probability.

Based on optimal control approach, guidance laws achieving a desired impact angle have been investigated in [1] and [2]. A generalized formulation of optimal guidance law was suggested to achieve an impact angle constraint and zero miss distance simultaneously for constant speed missiles in [1]. In [2], as a further study, the energy minimum cost function weighted by a power of the time-to-go was applied. By adjusting the exponent of the weighting function, guidance gains and trajectory shaping can be obtained.

Apart from optimal control based approach, Tahk et al. [3]-[7] have proposed guidance laws, which have simple polynomial form of time-to-go with unknown coefficients. The concept of the time-to-go polynomial guidance (TPG) has been firstly suggested for an automatic UAV landing in [3]. In many cases, such as proportional navigation guidance (PNG), derivatives of the PNG and optimal guidance laws, guidance command is expressed as a function of time-to-go. In [3], the authors firstly assumed that a guidance law would be a polynomial function of time-to-go and found coefficients satisfying terminal constraints. As a result, they noticed that the result was identical to the optimal guidance law considering zero miss distance and impact angle. Starting with [3], many researchers have been developing TPG. The TPG considering terminal impact angle and acceleration constraints have studied in [4], [5]. At the terminal homing phase, missile flight path angle and attitude should be aligned to maximize the warhead effect. However, in the case of maneuvering missile, attitude angle of the body is not aligned with flight path angle. In [4] and [6], a simple TPG is modified to satisfy zero miss distance, terminal impact angle and zero terminal acceleration constraints. Observability enhancement of TPG have been derived in [5] by introducing an oscillatory trajectory perpendicular to the collision course. The work in [7] suggested a polynomial guidance with additional bias term satisfying not only impact angle but also impact time constraints.

While a number of studies for the constant velocity missiles with impact angle constraint are developed, several guidance laws for missiles of varying velocity have been studied. Cho, Ryoo, and Tahk [8] investigated a closed-form solution of a time-to-go like function and the time-varying guidance gain. They found that the result can be expressed as the same form for constant velocity model. On the other hand, the authors of [9] suggested an energy minimizing solution for missiles of time-varying velocity under the impact angle constraint while the target maneuvers. Baba, Takehira, and Takano [10] studied a guidance law for velocity-varying missile against constant target maneuver. The author derived three guidance commands – the basic guidance equation, taking into account a missile thrust, and after thrust cutoff.

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Previous studies on TPG mentioned above are formulated under the assumption of constant velocity missiles. However, the missiles are losing their energy during homing phase because of aerodynamic drag and external disturbances. Especially, under the supersonic conditions, the velocity of missiles exponentially decreases as a function of time. The main contribution of this study is to expand the concept of conventional TPG law to a missiles with time-varying velocity. Two unknown coefficients of TPG are adopted, which are used to satisfy desired impact angle and zero miss distance.

This paper is organized as follows: in Section II, problem definition and engagement kinematics are introduced. Also, derivation of the time-to-go polynomial guidance law and how to determine the two coefficients are described. The numerical simulations are illustrated in Section III and finally, conclusion of this paper is stated in Section IV.

II. DERIVATION OF THE GUIDANCE LAW

A. Kinematics of System

This section deals with a problem formulation. We consider two-dimensional engagement of a homing missile M against a stationary target T. The planar engagement geometry is illustrated in Fig. 1. The missile flight path angle and line-of-sight angle are denoted as γ and σ , respectively. Relative distance between the missile and the target is expressed as R. The derivation of the guidance law and the kinematics are expressed in the impact angle frame. The impact angle frame is defined as the rotated from the inertial reference frame (X_I, Y_I) with terminal impact angle γ_f . The flight path angle with respect to the impact angle frame is defined as

$$\overline{\gamma} = \gamma - \gamma_f \tag{1}$$

In this paper, we assume that the velocity of the missile is not a constant. The velocity of the vehicle is decelerated by aerodynamic drag. For a high speed vehicle which is in a supersonic region, deceleration is proportional to the current velocity. Therefore, time derivative of the missile velocity can be expressed as



Figure 1. Planar engagement geometry.

$$\dot{V}(t) = -kV(t) \tag{2}$$

where the k is a decelerating parameter and typical range of the k is from 0.1 to 0.125 [11]. By integrating (2), velocity of the missile can be expressed as a function of time such as

$$V(t) = V_0 e^{-kt}$$

= $V_c e^{kt_{go}}$ (3)

where V_0 and V_f are the initial homing velocity and the terminal velocity, respectively and t_{go} is time-to-go which is the remaining time to the terminal impact moment. The time-to-go is defined as

$$t_{go} = t_f - t \tag{4}$$

For the nonlinear engagement kinematics, the missile acceleration command is denoted as a and its direction is perpendicular to the velocity vector of the missile, V(t). Therefore, the time derivative of $\overline{\gamma}$ is expressed as

$$\dot{\overline{\gamma}} = \frac{a}{V(t)} \tag{5}$$

Also, the relative position in the impact angle frame (x, y) are expressed as

$$\dot{x} = V(t)\cos\overline{\gamma}$$

$$\dot{y} = V(t)\sin\overline{\gamma}$$
(6)

If we assume that the mid-course phase is properly guided and initial homing flight path angle is close enough to the desired impact angle γ_f , then substituting (3) into (6) can be approximated as

$$\dot{x} = V_f e^{kt_{go}}$$

$$\dot{y} = V_e e^{kt_{go}} \overline{\gamma}$$
(7)

B. Derivation of the Guidance LAW

In this study, we initially assume that the guidance commands are the polynomial function of time-to-go, which is the same form with other literature [1]-[5]

$$a(t) = c_m t_{go}^m + c_n t_{go}^n \quad \text{where} \quad 0 < m < n \tag{8}$$

where the m and n are arbitrary positive integers and the coefficients c_m and c_n are unknown variables. In order to intercept the target, the coefficients should be calculated properly. By substituting (3) and (8) into (5), the flight path angle $\bar{\gamma}$ can be rewritten as

$$\overline{\gamma}(t) = \int_{0}^{t} \frac{1}{V_{0}e^{-kt}} \left(c_{m}t_{go}^{m} + c_{n}t_{go}^{n} \right) dt + \overline{\gamma}_{0}$$

$$= \int_{t_{go}}^{t_{f}} \frac{e^{kt_{go}}}{V_{f}} \left(c_{m}t_{go}^{m} + c_{n}t_{go}^{n} \right) dt_{go} + \overline{\gamma}_{0}$$
(9)

After some mathematical calculations, the results of (9) can be expressed as a function of time-to-go such that

$$\overline{\gamma}(t) = -\frac{c_m}{V_f} \frac{1}{k^{m+1}} \left\{ e^{-kt_f} B_m^{t_f} - e^{-kt_{go}} B_m^{t_{go}} \right\}$$

$$-\frac{c_n}{V_f} \frac{1}{k^{n+1}} \left\{ e^{-kt_f} B_n^{t_f} - e^{-kt_{go}} B_n^{t_{go}} \right\} + \overline{\gamma}_0$$
(10)

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where

$$B_{i}^{t} = k^{i} t^{i} + i k^{i-1} t^{i-1} + i (i-1) k^{i-2} t^{i-2} + \dots + i (i-1) \dots 2kt + i!$$

Substitute (10) into (7) and integrate \dot{y} with respect to time-to-go, then the relative position in the impact angle frame y can be determined as

$$y(t) = \frac{c_m}{k^{m+2}} \left\{ \frac{1}{m+1} k^{m+1} t_f^{m+1} - m! + e^{-kt_f} e^{kt_{go}} B_m^{t_f} \right\}$$
$$- \frac{c_m}{k^{m+2}} \frac{1}{m+1} \left\{ B_{m+1}^{t_{go}} - (m+1)! \right\}$$
$$\frac{c_n}{k^{n+2}} \left\{ \frac{1}{n+1} k^{n+1} t_f^{n+1} - n! + e^{-kt_f} e^{kt_{go}} B_n^{t_f} \right\}$$
(11)
$$- \frac{c_n}{k^{n+2}} \frac{1}{n+1} \left\{ B_{n+1}^{t_{go}} - (n+1)! \right\}$$
$$+ \frac{\overline{\gamma}_0 V_f}{k} \left\{ e^{kt_f} - e^{kt_{go}} \right\} + y_0$$

The lateral miss distance and the final flight path angle are determined by substituting $t = t_f$ or $t_{go} = 0$ into (11) and (10), respectively. In order to satisfy the zero miss distance and desired terminal impact angle, the following equation should be satisfied

$$\overline{\gamma}_{M}(t_{f}) = 0$$

$$y(t_{f}) = 0$$
(12)

The coefficients c_m and c_n can be determined by solution of (12) such as

$$c_{m} = \frac{1}{\det A} \times \left\{ \frac{\overline{\gamma}_{0}}{k^{n+2}} \frac{1}{n+1} \left(B_{n+1}^{t_{f}} - (n+1)! e^{kt_{f}} \right) + \frac{y_{0} e^{kt_{go}}}{Vk^{n+1}} \left(e^{-kt_{f}} B_{n}^{t_{f}} - n! \right) \right\}$$
(13)

$$c_{n} = \frac{1}{\det A} \times \left\{ -\frac{\overline{\gamma}_{0}}{k^{m+2}} \frac{1}{m+1} \left(B_{m+1}^{t_{f}} - (m+1)! e^{kt_{f}} \right) -\frac{y_{0} e^{kt_{go}}}{V k^{m+1}} \left(e^{-kt_{f}} B_{m}^{t_{f}} - m! \right) \right\}$$
(14)

where

$$\det A = \frac{e^{kt_f}}{V_0(m+1)(n+1)} \times \left\{ \frac{m+1}{k^{m+2}} \left(e^{-kt_f} B_m^{t_f} - m! \right) - \frac{n+1}{k^{n+2}} \left(e^{-kt_f} B_n^{t_f} - n! \right) \right\}$$
(15)

If we initialize and recalculate the coefficients of c_m and c_n at each time step, t_f can be replaced with t_{go} in (13), (14) and (15). The resultant guidance commands that satisfy the terminal constraints can be obtained by substituting (13) and (14) into (8). To implement the guidance law into the computer simulation, the coefficients are calculated numerically.

When k goes to zero, the following equations are held by L'hospital's rule.

$$\lim_{k \to 0} \frac{e^{-kt_f} B_m^{t_f} - (m)!}{k^{m+1}} = \frac{1}{m+1} t_f^{m+1}$$

$$\lim_{k \to 0} \frac{e^{-kt_f} B_{m+1}^{t_f} - (m+1)!}{k^{m+2}} = \frac{1}{n+2} t_f^{m+2}$$

$$\lim_{k \to 0} \frac{e^{-kt_f} B_n^{t_f} - (n)!}{k^{n+1}} = \frac{1}{n+1} t_f^{n+2}$$

$$\lim_{k \to 0} \frac{e^{-kt_f} B_{n+1}^{t_f} - (n+1)!}{k^{n+2}} = \frac{1}{n+2} t_f^{n+2}$$
(16)

Hence, we can verify that (13) and (14) are identical to the TPG for constant velocity if k is zero.

C. Time-to-go Estimation

The resultant form of the acceleration commands are in terms of time-to-go, therefore, time-to-go estimation is required. In the literature [7], approximated arc length of trajectory produced by the impact angle control is expressed as

$$S \approx R \left[1 + \frac{1}{15} \left(\lambda^2 + \lambda_f^2 - \frac{1}{2} \lambda \lambda_f \right) \right]$$
(17)

In this study, exact arc length cannot be calculated, because of complexity of (10). Therefore we considered (17) as the approximated arc length of trajectory for velocity-varying missiles. Then, arc length *S* should be equal to the integrating result of (3) from 0 to t_{go} , therefore time-to-go estimation for the missiles of time-varying velocity can be expressed as

$$\hat{t}_{go} = -\frac{1}{k} \ln \left[1 - \frac{k}{V} S \right]$$
(18)

Equation (18) is used for an estimated time-to-go value of the following simulations.

III. SIMULATION RESULTS

In this section, the performance of the proposed guidance law is investigated through numerical simulations with various guidance gains and terminal constraints. The gain set of the guidance law and homing conditions are shown in Table I. The initial position of the missile and target are (0,0) and (5000,8000), respectively. The initial velocity of the missile is 1500m/s. The non-linear simulations are conducted with lag-free acceleration and the acceleration limit of 50g ($g = 9.8m/s^2$). It may be a reasonable value for supersonic missiles. Moreover, the effect of the saturated acceleration is not a main issue of this study. The decelerating parameter is set as 0.01.

In order to compare the performance of proposed guidance law with respect to the typical TPG, which is designed for a missile of constant velocity, the guidance law [4] is used for comparison.

Note that time-to-go polynomial guidance for constant velocity missile is represented as TPG_C and time-to-go polynomial guidance for velocity varying missile, which is proposed in this paper, is expressed as TPG_V. The gain set (m,n) of TPG_C and TPG_V are (1,2), (1,3) and (2,3). Here, TPG_X-mn denotes the guidance law with a gain set (m,n).

The intercept trajectories of TPG_V-12 are shown in Fig. 2. For the desired impact angle with range of 0 to 90 degree for every 15 degree, the proposed guidance law leads the missiles to the correct impact courses. Fig. 3 shows the time histories of guidance commands for TPG_V-12 with various desired impact angles.

Values
(0,0) m
(5000, 8000) m
1500 m/s
90°
$0^{\circ} < \gamma_f < 90^{\circ}$ for every
15°
(1,2), (1,3), (2,3)
a < 50g
0.01

TABLE I. HOMING CONDITIONS FOR THE SIMULATIONS



Figure 2. Intercept Trajectories for TPG_V-12 with various desired impact angles



Figure 3. Time histories of guidance commands for TPG_V-12 with various desired impact angles.



Figure 4. Time histories of flight path angles for TPG_V-12 with various desired impact angles.

As it shows, TPG_V generates large guidance commands at early part of the engagement. As the missiles approach to the target, the terminal guidance commands converge to zero. The acceleration commands have been generated form of polynomial of time-to-go, therefore, the commands should converge to zero while time-to-go goes zero at the terminal time. Fig. 4 shows that the desired impact angle constraints are satisfied for all cases.



Figure 5. Intercept Trajectories for TPG_V with various guidance gain set.



Figure 6. Time histories of guidance commands for TPG_V with various guidance gain set.



Figure 7. Time histories of flight path angles for TPG_V with various guidance gain set.

In order to compare the characteristics of difference guidance gain set, TPG_V-12, TPG_V-13, and TPG_V-23 are simulated with same desired impact angle, $\gamma_f = 0^{\circ}$ and same initial flight path angel, $\gamma_0 = 90^{\circ}$. Fig. 5 shows

the trajectories of each guidance law. Compared to the trajectory of TPG_V-12, trajectories of TPG_V-13 and TPG_V-23 are stretched. By longer trajectories, TPG_V-23 needs longer engagement time as shown in Fig. 6.



Figure 8. Intercept Trajectories for TPG_V and TPG_C with various desired impact angles and various gain set.



Figure 9. Time histories of guidance commands for TPG_V and TPG_C with various desired impact angles and various gain set.



Figure 10. Time histories of flight path angles for TPG_V and TPG_C with various desired impact angles and various gain set.

The guidance commands of TPG_V-13 is larger than that of TPG_V-12, however, it rapidly approaches zero compared with TPG_V-12. These characteristics are identical with TPG_C as already known in [4]. Fig. 7 shows the time histories of flight path angles of TPG_V with various guidance gain set. For the comparison of the TPG V to the TPG C, the simulation is conducted with gain set of (1,2), (1,3), and (2,3) for TPG V and TPG C. Fig. 8 shows the trajectories of each guidance law. Also, the time histories of guidance commands are shown in Fig. 9. Finally, Fig. 10 illustrates the time histories of flight path angles. As mentioned in [8], TPG_C is identical to the optimal guidance law with the minimum energy cost weighted by time-to-go only if n = m+1. Therefore, we expect that TPG V-12 and TPG V-23 are also one of the optimal solution of time-to-go weighted cost function. In this study, we have not proved the optimality of TPG_V but the sub-optimality of TPG_V is investigated.

When we consider following equation as a cost function for m=1 or 2.

$$\min J_m = \int \frac{a^2}{t_{go}^m} \tag{19}$$

The cost ratio of each guidance law is shown in Table II. It shows that the TPG_V requires less time-to-go weighted cost during the engagement. The cost of TPG_V is scaled by TPG_C with same guidance gain and same terminal constraint. Although TPG_V may not be an optimal solution of minimizing the cost function (19), it is obvious that TPG_V is sub-optimal solution of the (19).

IV. CONCLUSION

In this paper, a time-to-go polynomial guidance (TPG) law satisfying impact angle constraints for missiles of time-varying velocity against stationary target is investigated. In order to expand the concept of conventional TPG law to a missiles with time-varying velocity, we assumed that the velocity of the missiles decreases with exponential form of time. The proposed guidance law includes two unknown coefficients which are used to satisfy desired impact angle and zero miss distance.

The solution of TPG for the time-varying velocity missiles is derived, and its coefficients are calculated numerically. If the decelerating parameter k is zero, the solution is exactly equal to the solution of conventional TPG. Numerical simulations are conducted to investigate the performance of suggested guidance law. For various guidance gain sets and scenarios, proposed guidance law successfully meets the terminal constraints.

Furthermore, time-to-go weighted energy costs are compared with typical TPG. The simulation results confirm that the proposed TPG is a sub-optimal solution of the minimizing time-to-go weighted energy. Its suboptimality is investigated by comparison to the conventional TPG in [4].

For the further study, closed-form solution of the acceleration commands can be investigated by direct substituting the coefficients into the time-to-go polynomial function. Then, more simple form of acceleration commands will be obtained. Also, proposed approach can be applied to the impact time control problems. In order to satisfy additional constraint such as impact time, guidance commands will be consist of more than three unknown coefficient. Also, more accurate estimation method of time-to-go should be studied.

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