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Research Paper

DETERMINATION OF CRACK PROPAGATION DIRECTION USING IP THEORY

Ashish K G Saran[1] and V G Ukadgaonker[2]

Classical design assumes homogeneous, defect free material. But all materials inherently contain some defects such as a hole, a notch, a slot, slag inclusions, cracks in weldment or heat affected zones due to uneven cooling, presence of foreign particles etc. So the assumption of continuum is not valid and the effect of these discontinuities (Parker A P, 1981) is to be considered. A crack may initiate and grow during the use of the component which after reaching a critical crack length leads to the failure of the component. Thus the problem of crack initiation and extension in the elasto-plastic region is important. Several theories have been proposed to describe the manner in which a crack will propagate in mixed mode condition such as Griffith's theory, Irwin's theory, MTS criterion, G-criterion, S-criterion, T-criterion, R-criterion etc. Adding to these a new theory of crack initiation, called *Ip theory* (Theocaris and Andriopoulos, 1982), was proposed by Ukadgaonker and Awasare. In this report an attempt has been made for the stress analysis during crack extension initiation using 'I_p theory of crack initiation' with basic objective of determination of angular location of crack extension initiation.

Keywords: Crack initiation, propagation, fracture mechanics, I_p theory, failure of component

INTRODUCTION

The total strain energy density is split in two components, namely, dilatational strain energy density (Papadopoulos, 1987) U_v and distortional strain energy density U_D. In two dimensional elastic problems these components are expressed as:

For plane stress,

expressed as:

For plane stress,

$$U_v = \left(\frac{1-2\nu}{6E}\right)(\sigma_x + \sigma_y)^2 \quad ...(1)$$

$$U_D = \left(\frac{1+\nu}{3E}\right)\left[(\sigma_x + \sigma_y)^2 - 3\left(\sigma_x\sigma_y - \tau_{xy}^2\right)\right] \quad ...(2)$$

And, or plane strain,

$$U_v = \frac{(1-2\nu)(1+\nu)^2}{6E}(\sigma_x + \sigma_y)^2 \quad ...(3)$$

1 Department of Mechanical Engineering, Quantum School of Technology, Roorkee.

2 Department of Mechanical Engineering, Indian Institute of Technology, Bombay.

$$U_D = \frac{(1-\nu)}{3E}[(\nu^2 - \nu + 1)(\sigma_x + \sigma_y)^2 - 3(\sigma_x\sigma_y - \tau_{xy}^2)] \quad ...(4)$$

Thus for plane stress,

$$U_v = \frac{(1-2\nu)}{2E}[I_1^2 - 2I_2] - \frac{(1-2\nu)}{(1+\nu)}U_D \quad ...(5)$$

And for plane strain,

$$U_v = \frac{(1-2\nu)(1+\nu)^2}{2E(1+2\nu-2\nu^2)}[I_1^2 - 2I_2] - \frac{(1-2\nu)(1+\nu)}{(1+2\nu-2\nu^2)}U_D \quad ...(6)$$

The square bracketed term in Equations (5) and (6) is defined as I_p which can be put in the form of an equation as follows:

$$I_p = I_1^2 - 2I_2 \quad ...(7)$$

Along the boundary of the elasto-plastic region U_D reaches its maximum value U_{Dmax}. Thus,

$$U_v = \frac{(1-2\nu)}{2E}I_p - \frac{(1-2\nu)}{(1+\nu)}U_{D_{max}} \quad ...(8)$$

for plane stress.

And,

$$U_v = \frac{(1-2\nu)(1+\nu)^2}{2E(1+2\nu-2\nu^2)}I_p - \frac{(1-2\nu)(1+\nu)}{(1+2\nu-2\nu^2)}U_{D_{max}} \quad ...(9)$$

for plane strain.

Equations (8) and (9) show that the dilatational strain energy density, U_v, along the boundary of the elasto-plastic boundary depends on the distortional strain energy density, U_D, and the term I_p. Thus for constant value of U_{Dmax}, i.e. along the elasto-plastic boundary, U_v reaches the maximum value when I_p becomes maximum. The maximum value of I_p gives the maximum value of U_v and the crack propagation takes place along that direction. Thus we get the direction of crack propagation given by the following conditions,

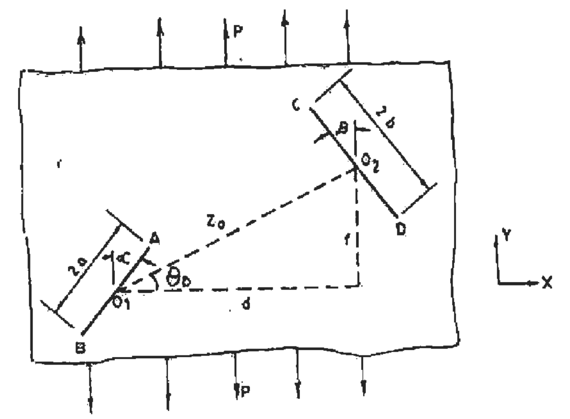
$$\frac{dI_p}{d\theta} = 0 \text{ and } \frac{d^2 I_p}{d\theta^2} < 0 \quad ...(10)$$

Equation (10) describes this new criterion.

Problem Statement

We have to find the direction of crack extension initiation for a system of two arbitrarily oriented cracks of different lengths subjected to uniform uniaxial tension at infinite (Ukadgaonker and Naik, 1991).

Figure 1: A Pair of Two Arbitrarily Oriented Cracks Subjected to Uniaxial Tension in an Isotropic Homogeneous Plate



Problem Formulation

The singular stress field in front of the crack tip for mixed mode condition is given by the following expressions:

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}}[\cos(\theta/2) - 1/2\sin\theta\cos(3\theta/2) - \mu\{2\sin(\theta/2) + \sin\theta\cos(3\theta/2)\}] \quad ...(11a)$$

or,

$$\sigma_x = \frac{K_I f_x(\theta)}{\sqrt{2\pi r}}$$

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}}[\cos(\theta/2) + 1/2\sin\theta\cos(3\theta/2) - (\mu/2)\sin\theta\cos(3\theta/2)] \quad ...(11b)$$

or,

$$\sigma_y = \frac{K_I f_y(\theta)}{\sqrt{2\pi r}}$$

And

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}}[1/2\sin\theta\cos(3\theta/2) + \mu(\cos(\theta/2) - 1/2\sin\theta\sin(3\theta/2))] \quad ...(11c)$$

or,

$$\tau_{xy} = \frac{K_I f_{xy}(\theta)}{\sqrt{2\pi r}}$$

Here (r, θ) are polar coordinates around the crack tip and,

$$\mu = \frac{K_{II}}{K_I}$$

where K_I and K_{II} are stress intensity factors in Mode I and Mode II respectively.

Substituting Equation (11) in term

$$I_p = [I_1^{\ 2} - 2I_2]$$

of Equations (7) or (8) we get,

$$I_p = \frac{1}{2\pi r}(c_{11}K_I^{\ 2} + c_{22}K_{II}^{\ 2} + c_{12}K_I K_{II}) \quad ...(12)$$

where,

$$c_{11} = 1/2(3 + 2\cos\theta - \cos^2\theta)$$

$$c_{22} = 1/2(3 - 2\cos\theta + 3\cos^2\theta)$$

$$c_{12} = 2\sin\theta(\cos\theta - 1)$$

Now the problem reduces to the finding of K_I and K_{II}.

K_I and K_{II} are given by the equations,

For tip A,

$$\begin{aligned} K_I = \sqrt{\frac{\pi}{a}}\frac{pb}{4}[2 &- \frac{B_{10}}{B_2^2}\cos(\theta_{B10} - 2\theta_{B2}) \\ &- \frac{B_{10}B_7}{B_4 B_5^3}\cos(\theta_{B10} + \theta_{B7} - \theta_{B4} - 3\theta_{B5}) \\ &+ \frac{2B_{10}B_6}{B_4^2 B_5^4}\cos(\theta_{B10} + \theta_{B6} - 2\theta_{B4} - 4\theta_{B5}) \\ &+ \frac{2B_{10}B_6}{B_4 B_5^4}\cos(\theta_{B10} + \theta_{B6} - \theta_{B4} - 4\theta_{B5}) \\ &- 2\cos 2\alpha + \frac{2}{B_5^2}\cos 2(\beta - \theta_{B5}) \\ &- \frac{2B_{11}}{B_8^2}\cos(\theta_{B11} - 2\theta_{B8}) \\ &- \frac{2B_{11}}{B_9^2}\cos(\theta_{B11} - 2\theta_{B9}) \\ &- \frac{2B_{10}}{B_5^2}\cos(2\theta_{B1} + \theta_{B10} - 2\theta_{B5}) \\ &+ \frac{2B_{10}}{B_5^3}\cos(2\theta_{B1} + \theta_{B10} - 3\theta_{B5})] \end{aligned} \quad ...(13)$$

$$K_{II} = \sqrt{\frac{\pi}{a}}\frac{pb}{4}[\frac{B_{10}}{B_2^2}\sin(\theta_{B10} - 2\theta_{B2})$$
$$+\frac{B_{10}B_7}{B_4B_5^3}\sin(\theta_{B10} + \theta_{B7} - \theta_{B4} - 3\theta_{B5})$$
$$-\frac{2B_{10}B_6}{B_4^2B_5^4}\sin(\theta_{B10} + \theta_{B6} - 2\theta_{B4} - 4\theta_{B5})$$
$$-\frac{2B_{10}B_6}{B_4B_5^4}\sin(\theta_{B10} + \theta_{B6} - \theta_{B4} - 4\theta_{B5})$$
$$-2\sin 2\alpha - \frac{2}{B_5^2}\sin 2(\beta - \theta_{B5})$$
$$+\frac{2B_{11}}{B_8^2}\sin(\theta_{B11} - 2\theta_{B8})$$
$$+\frac{2B_{11}}{B_9^2}\sin(\theta_{B11} - 2\theta_{B9})$$
$$+\frac{2B_{10}}{B_5^2}\sin(2\theta_{B1} + \theta_{B10} - 2\theta_{B5})$$
$$-\frac{2B_{10}}{B_5^3}\sin(2\theta_{B1} + \theta_{B10} - 3\theta_{B5})] \quad ...(14)$$

For tip B,

$$K_I = \sqrt{\frac{\pi}{a}}\frac{pb}{4}[2 - \frac{B_9}{B_3^2}\cos(\theta_{B9} - 2\theta_{B3})$$
$$-\frac{B_{10}B_7}{B_4^3B_5}\cos(\theta_{B10} + \theta_{B7} - 3\theta_{B4} - \theta_{B5})$$
$$+\frac{2B_{10}B_6}{B_4^4B_5^2}\cos(\theta_{B10} + \theta_{B6} - 4\theta_{B4} - 2\theta_{B5})$$
$$+\frac{2B_{10}B_6}{B_4^4B_5}\cos(\theta_{B10} + \theta_{B6} - 4\theta_{B4} - \theta_{B5}) -$$
$$-2\cos 2\alpha + \frac{2}{B_4^2}\cos 2(\beta - \theta_{B4})$$
$$-\frac{2B_{11}}{B_{13}^2}\cos(\theta_{B11} - 2\theta_{B13})$$
$$-\frac{2B_{11}}{B_{12}^2}\cos(\theta_{B11} - 2\theta_{B12})$$
$$-\frac{2B_{10}}{B_4^2}\cos(2\theta_{B1} + \theta_{B10} - 2\theta_{B4})$$
$$+\frac{2B_{10}}{B_4^3}\cos(2\theta_{B1} + \theta_{B10} - 3\theta_{B4})] \quad ...(15)$$

$$K_{II} = \sqrt{\frac{\pi}{a}}\frac{pb}{4}[\frac{B_9}{B_3^2}\sin(\theta_{B9} - 2\theta_{B3})$$
$$+\frac{B_{10}B_7}{B_4^3B_5}\sin(\theta_{B10} + \theta_{B7} - 3\theta_{B4} - \theta_{B5})$$
$$-\frac{2B_{10}B_6}{B_4^4B_5^2}\sin(\theta_{B10} + \theta_{B6} - 4\theta_{B4} - 2\theta_{B5})$$
$$-\frac{2B_{10}B_6}{B_4^4B_5}\sin(\theta_{B10} + \theta_{B6} - 4\theta_{B4} - \theta_{B5})$$
$$+2\sin 2\alpha - \frac{2}{B_4^2}\sin 2(\beta - \theta_{B4})$$
$$+\frac{2B_{11}}{B_{13}^2}\sin(\theta_{B11} - 2\theta_{B13})$$
$$+\frac{2B_{11}}{B_{12}^2}\sin(\theta_{B11} - 2\theta_{B12})$$
$$+\frac{2B_{10}}{B_4^2}\sin(2\theta_{B1} + \theta_{B10} - 2\theta_{B4})$$
$$-\frac{2B_{10}}{B_4^3}\sin(2\theta_{B1} + \theta_{B10} - 3\theta_{B4})] \quad ...(16)$$

For tip C,

$$K_I = \sqrt{\frac{\pi}{b}}\frac{pa}{4}[2 + \frac{A_8}{A_3^2}\cos(\theta_{A8} - 2\theta_{A3})$$
$$+\frac{A_7A_8}{A_4^3A_5}\cos(\theta_{A7} + \theta_{A8} - 3\theta_{A4} - \theta_{A5})$$
$$-\frac{2A_6A_8}{A_4^4A_5^2}\cos(\theta_{A6} + \theta_{A8} - 4\theta_{A4} - 2\theta_{A5})$$
$$-\frac{2A_6A_8}{A_4^4A_5}\cos(\theta_{A6} + \theta_{A8} - 4\theta_{A4} - \theta_{A5})$$

$$-2\cos 2\beta + \frac{2}{A_4^2}\cos 2(\alpha + \theta_{A4})$$
$$-\frac{2A_9}{A_{10}^2}\cos(\theta_{A9} - 2\theta_{A10})$$
$$-\frac{2A_9}{A_{11}^2}\cos(\theta_{A9} - 2\theta_{A11})$$
$$+\frac{2A_8}{A_4^2}\cos(2\theta_{A1} + \theta_{A8} - 2\theta_{A4})$$
$$-\frac{2A_8}{A_4^3}\cos(2\theta_{A1} + \theta_{A8} - 3\theta_{A4})] \quad ...(17)$$

$$K_{II} = \sqrt{\frac{\pi}{b}}\frac{pa}{4}[-\frac{A_8}{A_3^2}\sin(\theta_{A8} - 2\theta_{A3})$$
$$-\frac{A_7 A_8}{A_4^3 A_5}\sin(\theta_{A7} + \theta_{A8} - 3\theta_{A4} - \theta_{A5})$$
$$-\frac{2A_6 A_8}{A_4^4 A_5^2}\sin(\theta_{A6} + \theta_{A8} - 4\theta_{A4} - 2\theta_{A5})$$
$$+\frac{2A_6 A_8}{A_4^4 A_5}\sin(\theta_{A6} + \theta_{A8} - 4\theta_{A4} - \theta_{A5})$$
$$+2\sin 2\beta + \frac{2}{A_4^2}\sin 2(\alpha + \theta_{A4})$$
$$+\frac{2A_9}{A_{10}^2}\sin(\theta_{A9} - 2\theta_{A10})$$
$$+\frac{2A_9}{A_{11}^2}\sin(\theta_{A9} - 2\theta_{A11})$$
$$-\frac{2A_8}{A_4^2}\sin(2\theta_{A1} + \theta_{A8} - 2\theta_{A4})$$
$$+\frac{2A_8}{A_4^3}\sin(2\theta_{A1} + \theta_{A8} - 3\theta_{A4})] \quad ...(18)$$

For tip D,

$$K_I = \sqrt{\frac{\pi}{b}}\frac{pa}{4}[2 + \frac{A_8}{A_2^2}\cos(\theta_{A8} - 2\theta_{A2})$$
$$+\frac{A_7 A_8}{A_5^3 A_4}\cos(\theta_{A7} + \theta_{A8} - \theta_{A4} - 3\theta_{A5})$$
$$-\frac{2A_6 A_8}{A_4^2 A_5^4}\cos(\theta_{A6} + \theta_{A8} - 2\theta_{A4} - 4\theta_{A5})$$
$$-\frac{2A_6 A_8}{A_5^4 A_4}\cos(\theta_{A6} + \theta_{A8} - \theta_{A4} - 4\theta_{A5})$$
$$-\frac{2A_9}{A_{13}^2}\cos(\theta_{A9} - 2\theta_{A13})$$
$$+\frac{2}{A_5^2}\cos 2(\alpha + \theta_{A5}) - 2\cos 2\beta$$
$$-\frac{2A_9}{A_{12}^2}\cos(\theta_{A9} - 2\theta_{A12})$$
$$+\frac{2A_8}{A_5^2}\cos(2\theta_{A1} + \theta_{A8} - 2\theta_{A5})$$
$$-\frac{2A_8}{A_5^3}\cos(2\theta_{A1} + \theta_{A8} - 3\theta_{A5})] \quad ...(19)$$

$$K_{II} = \sqrt{\frac{\pi}{b}}\frac{pa}{4}[-\frac{A_8}{A_2^2}\sin(\theta_{A8} - 2\theta_{A2})$$
$$-\frac{A_7 A_8}{A_5^3 A_4}\sin(\theta_{A7} + \theta_{A8} - \theta_{A4} - 3\theta_{A5})$$
$$+\frac{2A_6 A_8}{A_4^2 A_5^4}\sin(\theta_{A6} + \theta_{A8} - 2\theta_{A4} - 4\theta_{A5})$$
$$+\frac{2}{A_5^2}\sin 2(\alpha + \theta_{A5}) - 2\sin 2\beta$$
$$+\frac{2A_6 A_8}{A_5^4 A_4}\sin(\theta_{A6} + \theta_{A8} - \theta_{A4} - 4\theta_{A5})$$
$$+\frac{2A_9}{A_{12}^2}\sin(\theta_{A9} - 2\theta_{A12})$$
$$+\frac{2A_9}{A_{13}^2}\sin(\theta_{A9} - 2\theta_{A13})$$
$$-\frac{2A_8}{A_5^2}\sin(2\theta_{A1} + \theta_{A8} - 2\theta_{A5})$$
$$+\frac{2A_8}{A_5^3}\sin(2\theta_{A1} + \theta_{A8} - 3\theta_{A5})] \quad ...(20)$$

Table 1: Direction of Crack Propagation at Various Crack Tips

Orientation of First Crack,		Orientation of the Line Connecting the Centers of the Two Cracks, Θ_0(degree)	Direction of Crack Propagation (degree)			
α(degree)	α(degree)		For tip A	For tip B	For tip C	For tip D
30	-30	0	65	85	60	81
30	-30	15	60	88	69	80

Various parameters used in the Equations (13) to (20) are described in Annexure 1. Once we have K_I and K_{II} around a crack tip we can find I_p using Equation (12). In this equation 'r' will have to be replaced by 'r_c' as we are interested in the value of I_p along the boundary of the elasto-plastic zone (Palaniswamy and Knauss, 1972) around the crack tip.

RESULTS

The directions of crack propagation are determined for two sets of values of parameters. The values are given in the Table 1.

CONCLUSION

As given above the directions of crack propagation are determined. The results conform to various analytical and experimental solutions for the same problem.

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ANNEXURE 1

$$A_1 = \frac{1}{a}[z_o^2 + a'^2 + 2z_o a' \cos(\theta_o - \theta_a)]^{\frac{1}{2}}$$

$$\theta_{A1} = \tan^{-1}\left(\frac{z_o \sin\theta_o + a' \sin\theta_a}{z_o \cos\theta_o + a' \cos\theta_a}\right)$$

$$a' = (z_o^4 + a^4 + 2z_o^2 a^2 \cos 2\theta_o)^{\frac{1}{4}}$$

$$\theta_a = \frac{1}{2}\tan^{-1}\left(\frac{z_o^2 \sin 2\theta_o}{z_o^2 \cos 2\theta_o - a^2}\right)$$

$$B_1 = \frac{1}{b}[z_o^2 + b'^2 + 2z_o b' \cos(\theta_o - \theta_b)]^{\frac{1}{2}}$$

$$\theta_{B1} = \tan^{-1}\left(\frac{z_o \sin\theta_o + b' \sin\theta_b}{z_o \cos\theta_o + b' \cos\theta_b}\right)$$

$$b' = (z_o^4 + b^4 - 2z_o^2 b^2 \cos 2\theta_o)^{\frac{1}{4}}$$

$$\theta_b = \frac{1}{2}\tan^{-1}\left(\frac{z_o^2 \sin 2\theta_o}{z_o^2 \cos 2\theta_o - b^2}\right)$$

$$A_2 = \sqrt{1 + A_1^2 + 2A_1 \cos\theta_{A1}}$$

$$\theta_{A2} = \tan^{-1}\left(\frac{A_1 \sin\theta_{A1}}{1 + A_1 \cos\theta_{A1}}\right)$$

$$A_3 = \sqrt{1 + A_1^2 - 2A_1 \cos\theta_{A1}}$$

$$\theta_{A3} = \tan^{-1}\left(\frac{A_1 \sin\theta_{A1}}{1 - A_1 \cos\theta_{A1}}\right)$$

$$A_4 = \sqrt{1 + A_1^2 + 2A_1 \cos\theta_{A1}}$$

$$\theta_{A4} = \tan^{-1}\left(\frac{-A_1 \sin\theta_{A1}}{1 + A_1 \cos\theta_{A1}}\right)$$

$$A_5 = \sqrt{1 + A_1^2 - 2A_1 \cos\theta_{A1}}$$

$$\theta_{A5} = \tan^{-1}\left(\frac{A_1 \sin\theta_{A1}}{1 - A_1 \cos\theta_{A1}}\right)$$

$$A_6 = \sqrt{1 + A_1^4 + 2A_1^2 \cos 2\theta_{A1}}$$

$$\theta_{A6} = \tan^{-1}\left(\frac{A_1^2 \sin\theta_{A1}}{1 - A_1^2 \cos 2\theta_{A1}}\right)$$

$$A_7 = \sqrt{9 + A_1^4 + 6A_1^2 \cos 2\theta_{A1}}$$

$$\theta_{A7} = \tan^{-1}\left(\frac{-A_1^2 \sin 2\theta_{A1}}{3 + A_1^2 \cos 2\theta_{A1}}\right)$$

$$A_8 = \sqrt{5 - 4\cos 2\alpha}$$

$$\theta_{A8} = \tan^{-1}\left(\frac{2\sin 2\beta - \sin 2(\alpha+\beta)}{2\cos 2\beta - \cos 2(\alpha+\beta)}\right)$$

$$A_9 = \sqrt{2 - 2\cos 2\alpha}$$

$$\theta_{A9} = \tan^{-1}\left(\frac{\sin 2\alpha}{1 - \cos 2\alpha}\right)$$

$$A_{10} = \sqrt{1 + A_4^2 + 2A_4 \cos(\alpha + \beta - \theta_{A4})}$$

$$\theta_{A10} = \tan^{-1}\left(\frac{A_4 \sin\theta_{A4} + \sin(\alpha+\beta)}{A_4 \cos\theta_{A4} + \cos(\alpha+\beta)}\right)$$

$$A_{11} = \sqrt{1 + A_4^2 - 2A_4 \cos(\alpha + \beta - \theta_{A4})}$$

$$\theta_{A11} = \tan^{-1}\left(\frac{A_4 \sin\theta_{A4} - \sin(\alpha+\beta)}{A_4 \cos\theta_{A4} - \cos(\alpha+\beta)}\right)$$

$$A_{12} = \sqrt{1 + A_5^2 + 2A_5 \cos(\alpha + \beta - \theta_{A5})}$$

$$\theta_{A12} = \tan^{-1}\left(\frac{A_5 \sin\theta_{A5} + \sin(\alpha+\beta)}{A_5 \cos\theta_{A5} + \cos(\alpha+\beta)}\right)$$

ANNEXURE 1 (CONT.)

$$A_{13} = \sqrt{1 + A_5^2 - 2A_5\cos(\alpha + \beta - \theta_{A5})}$$

$$\theta_{A13} = \tan^{-1}\left(\frac{A_5\sin\theta_{A5} - \sin(\alpha+\beta)}{A_5\cos\theta_{A5} - \cos(\alpha+\beta)}\right)$$

$$B_2 = \sqrt{1 + B_1^2 + 2B_1\cos\theta_{B1}}$$

$$\theta_{B2} = \tan^{-1}\left(\frac{-B_1\sin\theta_{B1}}{1 - B_1\cos\theta_{B1}}\right)$$

$$B_3 = \sqrt{1 + B_1^2 + 2B_1\cos\theta_{B1}}$$

$$\theta_{B3} = \tan^{-1}\left(\frac{B_1\sin\theta_{B1}}{1 + B_1\cos\theta_{B1}}\right)$$

$$B_4 = \sqrt{1 + B_1^2 + 2B_1\cos\theta_{B1}}$$

$$\theta_{B4} = \tan^{-1}\left(\frac{-B_1\sin\theta_{B1}}{1 + B_1\cos\theta_{B1}}\right)$$

$$B_5 = \sqrt{1 + B_1^2 - 2B_1\cos\theta_{B1}}$$

$$\theta_{B5} = \tan^{-1}\left(\frac{B_1\sin\theta_{B1}}{1 - B_1\cos\theta_{B1}}\right)$$

$$B_6 = \sqrt{1 + B_1^4 + 2B_1^2\cos 2\theta_{B1}}$$

$$\theta_{B6} = \tan^{-1}\left(\frac{-B_1^2\sin 2\theta_{B1}}{1 + B_1^2\cos 2\theta_{B1}}\right)$$

$$B_7 = \sqrt{9 + B_1^4 + 6B_1^2\cos 2\theta_{B1}}$$

$$\theta_{B7} = \tan^{-1}\left(\frac{-B_1^2\sin 2\theta_{B1}}{3 + B_1^2\cos 2\theta_{B1}}\right)$$

$$B_8 = \sqrt{1 + B_5^2 + 6B_5\cos(\alpha + \beta + \theta_{B5})}$$

$$\theta_{B8} = \tan^{-1}\left(\frac{B_4\sin\theta_{B4} - \sin(\alpha+\beta)}{B_4\cos\theta_{B4} + \cos(\alpha+\beta)}\right)$$

$$B_9 = \sqrt{1 + B_5^2 - 2B_5\cos(\alpha + \beta + \theta_{B5})}$$

$$\theta_{B9} = \tan^{-1}\left(\frac{B_4\sin\theta_{B4} + \sin(\alpha+\beta)}{B_4\cos\theta_{B4} - \cos(\alpha+\beta)}\right)$$

$$B_{10} = \sqrt{5 - 4\cos 2\beta}$$

$$\theta_{B10} = \tan^{-1}\left(\frac{2\sin 2\alpha - \sin 2(\alpha+\beta)}{2\cos 2\alpha - \cos 2(\alpha+\beta)}\right)$$

$$B_{11} = \sqrt{2 - 2\cos 2\beta}$$

$$\theta_{B11} = \tan^{-1}\left(\frac{\sin 2\beta}{\cos 2\beta - 1}\right)$$