Comparisons of Break Points Selection Strategies for Piecewise Linear Approximation

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Abstract—**Many optimization approaches adopt piecewise linear functions to solve real applications that are formulated as nonlinear programming problems. The number of the break points and the positions of the break points are two major factors that affect the quality of the linear approximation. Most of existing methods select evenly-spaced break points for constructing a piecewise linear approximation of a nonlinear function. This study investigates the impact of different break points selection strategies on the accuracy of the linear approximation. Two numerical experiments are presented to compare the performance of different break points selection strategies in solving nonlinear programming problems.**

Index Terms—**global optimization, break point selection, piecewise linear function.**

I. Introduction

Piecewise linear functions (PLFs) are frequently used to solve real applications that are formulated as nonlinear programming problems, such as engineering design, inventory control, production planning, and portfolio management *etc*. Much research has discussed how to piecewise linearize a nonlinear function as a mixed-integer program in last few decades. The commonly used textbooks (Bazaraa *et al.* [1], Taha [2]) of nonlinear programming provide some methods to formulate PLFs. Recently various mixed-integer programming models for PLFs have been proposed by Kontogiorgis [3], Padberg [4], Croxton *et al.* [5], Keha *et al.* [6], Li *et al.* [7], Vielma and Nemhauser [8]. For expressing a piecewise linear function of a single variable x with $m+1$ break points (i.e., m line segments), most methods mentioned above require additional m binary variables and $4m$ constraints. Li *et al.* [7] developed a representation method for PLFs by using $\lceil \log_2(m-1) \rceil$ additionalbinaryvariables and $8+8\lceil \log_2(m-1) \rceil$ additional constraints. Vielma and Nemhauser [8] developed another logarithmic method for piecewise linearizing functions of one and two variables.

The number of the break points and the positions of the break points are two major factors that affect the quality of linear approximation. Most of the above linearization methods use numerous evenly-spaced break points for constructing a piecewise linear approximation of a nonlinear function with a low error. Since adding numerous break points substantially increases the number of additional variables and constraints required for expressing PLFs, some research investigated how to select the break points to accelerate the process of finding the optimal solution. Meyer [9] and Bazaraa *et al.* [1] generated finer break points around the obtained optimal solution computed by the previous problem. Li and Yu [10] selected finer break point at maximal error of linear approximation. Kontogiorgis [3] placed the break points uniformly or minimizing the approximation error. Lundell [11] discussed three break points selection strategies: (i) the solution point of the previous iteration; (ii) the midpoint of the interval of existing break points; (iii) the point with largest approximation error.

Both Li *et al.* [7] and Vielma and Nemhauser [8] developed logarithmic methods for piecewise linearizing a non-linear function. The computational results in Vielma and Nemhauser [8] show that their piecewise linearization technique outperforms other piecewise linearization formulations including the SOS2 (special ordered sets of type 2) model without binary variables. This study employs the Vielma and Nemhauser [8] method to piecewise linearize nonlinear functions with three different break points selection strategies. Several experiments are conducted to compare the performance of different break points selection strategies in solving nonlinear programming problems.

II. Piecewise Linear Techniques

Consider a nonlinear function $f(x)$ where $a_0 \le x \le a_m$, and denote $a_1, a_2, \cdots, a_{m-1}$ being other break points between a_0 and a_m for representing the piecewise linear function of $f(x)$, where $a_0 < a_1 < a_2 < \cdots < a_{m-1} < a_m$. Let $P = \{0,1,2,\ldots,m\}$ and $p \in P$. An injective function for modelling a piecewise linear functions is described as follows.

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Remark 1: An injective function $B:\{1,2,\ldots,m\}\rightarrow\{0,1\}^{\theta}$, $\theta=\lceil\log_2 m\rceil$, where the vectors $B(p)$ and $B(p+1)$ differ in at most one component for all $p\in\{1,2,\ldots,m-1\}$, can always be constructed [8].

Let $B(p)=(u_1,u_2,\ldots,u_\theta)$, $\forall u_k\in\{0,1\}$, $k=1,2,\ldots,\theta$, and $B(0)=B(1)$. Some notations are introduced below:

$S^+(k)$: a set composed of all p where $u_k=1$ of $B(p)$ and $B(p+1)$ for $p=1,2,\ldots,m-1$, or $u_k=1$ of $B(p)$ for $p\in\{0,m\}$, *i.e.*,

$S^+(k)=\{p\mid\forall B(p)\text{ and }B(p+1),u_k=1,p=1,2,...,m-1\}\cup\{p\mid\forall B(p),u_k=1,p\in\{0,m\}\}$.

$S^-(k)$: a set composed of all p where $u_k=0$ of $B(p)$ and $B(p+1)$ for $p=1,2,\ldots,m-1$, or $u_k=0$ of $B(p)$ for $p\in\{0,m\}$, *i.e.*,

$S^-(k)=\{p\mid\forall B(p)\text{ and }B(p+1),u_k=0,p=1,2,...,m-1\}\cup\{p\mid\forall B(p),u_k=0,p\in\{0,m\}\}$.

Tsai and Lin [12] deduced the following theorem to construct a linear approximate of a nonlinear function by the Vielma and Nemhauser [8] technique.

Theorem 1: Given a univariate function $f(x)$, $a_0\le x\le a_m$, denote $L(f(x))$ as the piecewise linear function of $f(x)$, where $a_0<a_1<a_2<\cdots<a_m$ be the $m+1$ break points of $L(f(x))$ can be expressed as

$$L(f(x))=\sum_{p=0}^{m}f(a_p)\lambda_p\ ,\quad x=\sum_{p=0}^{m}a_p\lambda_p\ ,\quad \sum_{p=0}^{m}\lambda_p=1\ ,$$

$$\sum_{p\in S^+(k)}\lambda_p\le u_k\ ,\quad \sum_{p\in S^-(k)}\lambda_p\le 1-u_k\ ,\quad \forall\lambda_p\in\Re_+\ ,\quad \forall u_k\in\{0,1\}.$$

III. Break Points Selection Strategies

Enough break points are required in the linearization process to construct a linear approximation of a nonlinear function with a low approximation error. However, adding numerous break points substantially increases the size of the reformulated problem and results in long solution time. This study investigates break points selection strategies to further enhance the computational efficiency in solving nonlinear programming problems. Three existing break points selection strategies [11] listed in the following are compared in this study.

1) Add a new break point at the midpoint of each interval of existing break points;
2) Add a new break point at the point with largest approximation error of each interval;
3) Add a new break point at the previously obtained solution point.

If a new break point is added at the midpoint of each interval of existing break points or at the point with largest approximation error, the number of line segments becomes double in each iteration. If a new break point is added at the previously obtained solution, one more line segment increases in each iteration.

IV. Numerical Examples

Two examples are presented to demonstrate the impact of different break points selection strategies on the accuracy of the linear approximation and the performance of solving nonlinear programming problems. All reformulated programs are solved by LINGO 11.0 on a PC with an Intel Core 2 Quad 2.66 GHz CPU and 3.46 GB Memory.

Example 1: The problem introduced in [7] is used to compare the accuracy of the linear approximation by different break points selection strategies

Minimize $x_1^{\alpha_1}-x_2^{\beta_1}$

subject to $x_1^{\alpha_2}-6x_1+x_2^{\beta_2}\le b,$

$x_1+x_2\le 8,$

$1\le x_1\le 7.4\,,1\le x_2\le 7.4,$

where $\alpha_1,\beta_1,\alpha_2,\beta_2,$ and b are fixed constants. By specifying three sets of different values for $\alpha_1,\beta_1,\alpha_2,\beta_2,b$ referred from [7], two problems are generated for comparing the performance of three break points selection strategies. In the first problem with ($\alpha_1,\beta_1,\alpha_2,\beta_2,b$) = (0.4,2,1.85,2,5), $x_1^{0.4}$ and $-x_2^2$ are concave terms required to be piecewise linearized, while the other convex terms $x_1^{1.85}$ and x_2^2 do not need linearization. The original problem becomes the following mixed-integer problem:

Minimize y_1-y_2

subject to $x_1^{1.85}-6x_1+x_2^2-5\le 0,$

$x_1+x_2-8\le 0,\ y_1=L(x_1^{0.4}),\ y_2=L(x_2^2),$

$1\le x_1\le 7.4\,,1\le x_2\le 7.4,$ (1)

where $L(x_1^{0.4})$ and $L(x_2^2)$ are piecewise linear functions of $x_1^{0.4}$ and x_2^2, respectively.

Table I lists number of line segments (m),accumulated CPU time, solution, objective of reformulated model, error in objective, and error in constraint in each iteration under three break points selection strategies with different numbers of line segments. The error in objective is evaluated by $\left|\ (x_1^*)^{0.4}-(x_2^*)^2-\left(L((x_1^*)^{0.4})-L((x_2^*)^2)\right)\right|$ and the error in constraint is evaluated by $Max\Big(Max\Big((x_1^*)^{1.85}-6x_1^*+(x_2^*)^2-5,\ x_1^*+x_2^*-8\Big),\ 0\Big)$ on the obtained solution (x_1^*,x_2^*). With the error in constraint below 10^{-5}, the error in objective decreases as the number of line segments increases. That is, the objective value approximates the real global objective value better as the number of break points increases. Selecting break points at midpoint or at the point of maximumapproximationerror does not obviously affect the performance of solving the nonlinear problem. However, adding a new break point at the previously obtained solution point spends less CPU time and obtains a solution closer to the real global solution than other two methods.

In the second problem with ($\alpha_1,\beta_1,\alpha_2,\beta_2,b$) = (0.5,0.5,0.8,0.9,-7) referred from Li *et al.* [7] , $x_1^{0.5}$, $x_1^{0.8}$, and $x_2^{0.9}$ are concave terms required to be piecewise linearized and $-x_2^{0.5}$ is convex. The original problem becomes the following convex mixed-integer nonlinear programming problem:

Minimize $y_1 - x_2^{0.5}$

subject to $y_2 - 6x_1 + y_3 - 5 \le 0$,

$x_1 + x_2 - 8 \le 0$,

$y_1 = L(x_1^{0.5})$, $y_2 = L(x_1^{0.8})$, $y_3 = L(x_2^{0.9})$

$1 \le x_1 \le 7.4$, $1 \le x_2 \le 7.4$,

where $L(x_1^{0.5})$, $L(x_1^{0.8})$ and $L(x_2^{0.9})$ are piecewise linear functions of $x_1^{0.5}$, $x_1^{0.8}$ and $x_2^{0.9}$, respectively. Since x_1 is involved in two piecewise linear functions and the points of the largest approximation error in the same interval for these two functions are different, this study only compares the methods of adding a new break point at the midpoint of each interval and adding a new break point at the previously obtained solution point. Table 2 lists experiment results of Example 1 with ($\alpha_1,\beta_1,\alpha_2,\beta_2,b$) = (0.5, 0.5, 0.8, 0.9, -7). Compared with the midpoint strategy, the previously solution point strategy has a faster speed of convergence to the real global optimal solution.

TABLE I. RESULTS OF EXAMPLE 1 WITH ($\alpha_1,\beta_1,\alpha_2,\beta_2,b$) = (0.4,2,1.85,2,5)

Iteration	m	Accumulated CPU time(ss:ms)	Solution(x_1^*, x_2^*)	Objective	Error in objective	Error in constraint
Break point selection: midpoint						
1	2	00:687	(3.701948,3.993769)	-14.912872	0.650680	$<10^{-5}$
2	4	00:905	(3.813995,3.997974)	-14.565576	0.290032	$<10^{-5}$
3	8	01:373	(3.843920,3.998755)	-14.399169	0.122729	$<10^{-5}$
4	16	02:028	(3.855708,3.999022)	-14.316734	0.040258	$<10^{-5}$
5	32	03:058	(3.850149,3.998899)	-14.276803	0.000321	$<10^{-5}$
6	64	04:477	(3.850863,3.998915)	-14.276625	0.000142	$<10^{-5}$
7	128	07:644	(3.852289,3.998947)	-14.276538	0.000053	$<10^{-5}$
8	256	15:881	(3.852457,3.998951)	-14.276511	0.000024	$<10^{-5}$
Break point selection: point of maximumapproximationerror						
1	2	00:109	(3.813229,3.997952)	-14.896051	0.620546	$<10^{-5}$
2	4	00:234	(3.837631,3.998603)	-14.565364	0.289017	$<10^{-5}$
3	8	00:484	(3.844592,3.998770)	-14.399441	0.123000	$<10^{-5}$
4	16	01:014	(3.856414,3.999038)	-14.316678	0.040200	$<10^{-5}$
5	32	01:857	(3.850687,3.998911)	-14.276781	0.000299	$<10^{-5}$
6	64	02:980	(3.851319,3.998925)	-14.276624	0.000143	$<10^{-5}$
7	128	05:242	(3.851625,3.998932)	-14.276546	0.000063	$<10^{-5}$
8	256	14:337	(3.852350,3.998948)	-14.276512	0.000030	$<10^{-5}$
Break point selection: previous solution point						
1	1	00:094	(3.849184,3.998877)	-24.644387	10.367909	$<10^{-5}$
2	2	00:250	(3.679977,3.992706)	-14.288065	0.030348	$<10^{-5}$
3	3	00:468	(3.912514,4000000)	-14.280144	0.005912	$<10^{-5}$
4	4	00:686	(3.853261,3.998969)	-14.276488	0.000001	$<10^{-5}$
5	5	01:810	(3.853261,3.998969)	-14.276487	$<10^{-6}$	$<10^{-5}$

TABLE II. RESULTS OF EXAMPLE 1 WITH ($\alpha_1,\beta_1,\alpha_2,\beta_2,b$) = (0.5,0.5,0.8,0.9, -7)

Iteration	m	Accumulated CPU time(ss:ms)	Solution(x_1^*, x_2^*)	Objective	Error in objective	Error in constraint
Break point selection: midpoint						
1	2	00:109	(2.274817,5.725183)	-0.974679	0.090193	0.089606
2	4	00:250	(2.286923,5.713077)	-0.897593	0.019646	0.016031
3	8	00:452	(2.288718,5.711282)	-0.882750	0.005772	0.005121
4	16	00:722	(2.289399,5.710601)	-0.877568	0.000958	0.000981
5	32	01:061	(2.289502,5.710498)	-0.876910	0.000355	0.000355
6	64	01:841	(2.289555,5.710445)	-0.876561	0.000035	0.000033
7	128	02:886	(2.289558,5.710442)	-0.876540	0.000015	0.000015
8	256	04:259	(2.289560,5.710440)	-0.876529	0.000006	$<10^{-5}$

Iteration	m	Accumulated CPU time(ss:ms)	Solution(x_1^*, x_2^*)	Objective	Error in objective	Error in constraint
Break point selection: previous solution point						
1	1	00:124	(2.254374,5.745626)	-1.059833	0.164287	0.213827
2	2	00:234	(2.288782,5.711218)	-0.880208	0.003265	0.004732
3	3	00:358	(2.289543,5.710457)	-0.876604	0.000071	0.000106
4	4	00:499	(2.289560,5.710440)	-0.876525	0.000002	$<10^{-5}$

V. CONCLUSIONS

Piecewise linear functions are frequently applied in many optimization methods for nonlinear problems. Most of existing methods select evenly-spaced break points for constructing a piecewise linear approximation of a nonlinear function. Since an appropriate selection of the break points can decrease the maximum tolerated distance between original nonlinear function and piecewise linear approximation, this study compares the impact of different break points selection strategies on the performance of solving nonlinear problems. More numerical experiments should be conducted and theoretical analysis should be done to investigate the break points selection strategies more completely.

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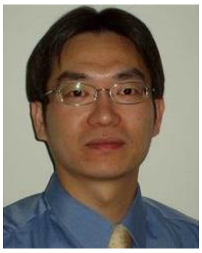

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