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Research Paper

FINITE ELEMENT VIBRATION ANALYSIS OF PRE-STRESSED FUNCTIONALLY GRADED PLATES

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The present theory is based on a Higher-Order displacement model and the three-dimensional Hooke's laws for plate material. The theory represents a more realistic quadratic variation of the transverse shearing and normal strains through the thickness of the plate. Nine-node Lagrangian elements have been used for the purpose of discretization using a refined Higher Order Shear deformation Theory (HOST12) that includes the effects of transverse shear deformations, transverse normal deformation and rotary inertia. A C0 isoperimetric finite element formulation is presented to calculate the required number of lowest natural frequencies of Functionally Graded Plates (Functionally graded plates) subjected to in-plane pre-stress. The material properties of the functionally graded plates are assumed to vary continuously from one surface to another, according to a simple power law distribution in terms of the constituent volume fractions. The formulation is applicable to thin as well as thick plates. The plate structure is idealized into an assemblage of nine-noded iso-parametric quadrilateral elements with Twelve degrees of freedom per node. Poisson's ratio has been assumed to be constant throughout the thickness and the material is assumed to be isotropic at a point. Hamilton's principle is used for the formulation. The effect of in-plane pre-stress is taken care by calculating the geometric stiffness matrix. The same shape functions are used to calculate the elastic stiffness matrix, geometric stiffness matrix and the element mass matrix. The consistent mass matrix is diagonalized by a special mass lumping scheme which conserves the total mass of the element and includes the effects due to rotary inertia terms. Subspace Iteration technique is applied to extract the natural frequencies. Numerical results for first seven natural frequencies are presented for rectangular and square plates under various boundary conditions. A parametric study has been carried out. Variations of natural frequencies with the constituent volume fractions are presented. The results show good agreement with three-dimensional analytical formulation.

Keywords: Vibration analysis-functional graded plate

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INTRODUCTION

Plates are structural elements whose thickness is very small compared to other two dimensions. In general, plates are loaded in transverse direction. They are the major load carrying members in most of the structures. Plates made out of Functionally Graded Material (FGM) have a gradual change in material properties within the plate as a function of position. The grading is done from metal to ceramic. Generally the gradation in plates is only in thickness direction. The gradient in properties is caused by the position dependent chemical composition. Extensive application of FGM plates is found in structural engineering. They are used in nuclear reactors, automobiles, spacecrafts, solar panels, thermoelectric instruments etc. An accurate prediction of behavior of plates not only improves the safety but also reduce the material cost. Whole of aircraft body is made of plates and shells stiffened by various types of extruded sections at suitable intervals in both directions. Most of the road vehicles and railway engines as well as wagons are the combination of stiffened plates and shells. Similarly boilers and machines are basically stiffened plates and shells. All these structures are subjected to different types of dynamic loads of varying time intervals causing structures to vibrate. The vibrations of the structures cause the reversible or at least variable stresses in various components. The variation of the stresses can cause fatigue failure. If there are any pre-existing flaws or cracks, their propagation also depends on the number of loading cycles and stress levels in each cycle. Hence it is extremely important to know the natural frequencies of vibration as well as response to the dynamic loads. Most of the structures referred above are subjected to the stresses due to static loads before being subjected to the dynamic loads. These stresses change the natural frequencies and response to the dynamic loads. Hence, it is necessary to take into account the pre-stress in the vibration analysis. The determination of the dynamic response of a plate structure or shell is a problem of immense practical significance in structural design. The evaluation of its free vibration response characteristics is a first step as a prelude to the prediction of the forced response of the system to any external excitations.

There are large varieties of forcing functions or excitations depending on the industry and working environment. Hence it is not advisable to include them in the work of general nature. Hence the present work is restricted to free vibration response of pre-stressed plates. The effect of pre-stress by in-plane forces is to increase or decrease the stiffness for out of plane deformation depending on whether the stress is tensile or compressive.

The Classical Plate Theories that are based on Kirchhoff hypothesis are inadequate to predict the gross response characteristics of moderately thick and thick composite plates or highly anisotropic composite plates. In order to get accurate gross responses from Mindlin type first order shear deformation theory of plates (the transverse shear strains areassumed to be constant in thickness direction), shear correction factors have to be incorporated to adjust the transverse shear stiffness for dynamic analysis of plates. Hence the accuracy of solution depends on the estimation of shear correction factors.

In Third order Shear Deformation Theory (TSDT), it requires no shear correction factors. The theory also contains the FOST as a special case. The displacement model that neglects the transverse normal stress effect and satisfying the zero transverse shear stress conditions on the bounding planes of the plate, thereby expressing the three displacements of a point in the plate space in terms of only five physical mid-plane displacement quantities. With this, the displacement model gives rise to second order derivatives of transverse displacement in the energy expression and hence displacement based finite element formulation requires C1 continuous shape functions which are computationally inefficient and are not amenable to the popular and widely used iso parametric formulation in the present day finite element technology. The higher order plate theory, HOST12 is based on the displacement model that contains cubic variation of displacements in each co-ordinate direction and the three dimensional Hooke's laws for plate material. The theory represents a more realistic quadratic variation of the transverse shearing and normal strains through the thickness of the plate. The theory does not require any shear correction factor and ultimately resulting in good estimation of shear strain energy. The formulation is applicable to thin as well as thick plates.

Nine-noded Lagrangian elements have been chosen for the purpose of analysis using a refined Higher Order Plate theory (HOST12) that includes the effects of transverse shear deformations, transverse normal deformation and rotary inertia. Hence, in this present work, the free vibration analysis of pre-stressed FG plates using HOST12 is considered, which is a gap found in the literature for vibration analyses.

LITERATURE REVIEW

Many researchers, till now, used a variety of methods to calculate the displacements and to perform vibration analysis; these methods include both analytical and numerical methods. Here a review of literature pertinent to the analysis of FGMs is carried out based on analytical and numerical developments of FG Plates.

Literature Based on FG Plates

FGMs can be classified based on grading direction and grading function. The FGM can be produced by continuously varying the constituents of multi-phase materials in a predetermined profile. An FGM can be defined by the variation in the volume fractions. Most of the researchers applied the following types of gradation functions for the material properties of FGMs.

Literature Based on Initial Stresses

Matsunga (2001) presented a global higher order theory to analyze the natural frequencies, modal displacements and stresses for cross-ply laminated composite plates subjected to initial in-plane stresses by taking into consideration the effects of shear, normal deformations and rotary inertia. He derived equations of motion by using Hamilton's principle. Natural frequencies were obtained by solving Eigenvalue problem numerically. The buckling stresses were computed by increasing the absolute values of compressive stresses until the natural frequency becomes zero. Yang *et al.* (2003) presented a large vibration analyses of pre-stressed FGM laminated plates based on Reddy's higher-order shear deformation plate theory.

Literature Based on Finite Elements

Fuchiyama *et al.* (1993) studied transient thermal stress behaviour of FGMs with cracks using an eight-node quadrilateral axi-symmetrical element. The analysis of FG plate including the effects of transverse shear strains, rotary inertia and moderately large rotations in the von-Karman theory was carried out by Praveen and Reddy (1998), in which plate finite element method was employed to investigate the static and dynamic responses of the FG plate by varying the volume fraction of the constituents. Han and Liu (2002) presented a computational method to investigate wave propagation in FG plates. The material properties were assumed as a quadratic function in the thickness direction. Della Croce and Venini (2004) presented a hierarchic family of finite elements for the analysis of Reissner-Mindlin FG plates based on variational formulation by assuming material properties to vary with the power law in terms of volume fractions of the constituent.

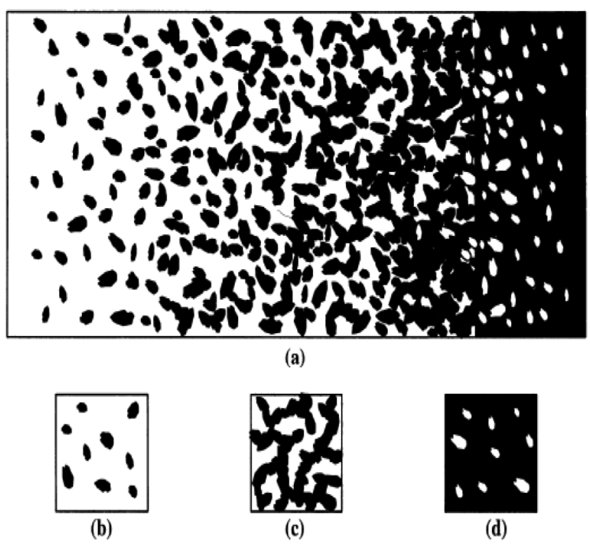
EXPERIMENTAL SETUP

Functionally Graded Material

FGMs possess smooth variation of mechanical properties such as modulus of elasticity, Poisson's ratio, coefficient of thermal expansion, tensile strength etc. The concept of FGM was proposed in 1984 by material scientists in Japan for preparing thermal barrier materials. The unique idea of a FGM was proposed to prepare a new composite by using heat-resistant ceramics on the high-temperature side and tough metals with high thermal conductivity on the low temperature side, with a gradual compositional variation from ceramic to metal. Therefore, FGMs are composite materials with a microscopically inhomogeneous character. Gradual changes in their microstructure distinguish FGMs from conventional composite materials. The continuous change in composition results in gradients in the properties of FGMs.

Structure: FGMs are heterogeneous composite materials made of two materials having entirely different characteristics. The microstructure of an FGM varies from non metal to metal as shown in Figure 1. Usually, the non metal will be ceramics like Zirconia, Silicon Carbide, Silicon Nitride and the metal will be Aluminium, Stainless Steel, etc.

Figure 1: The Microstructure of an FGM Varies from Non-Metal to Metal



Note: (a) Shows the gradual and smooth variation of constituent volume fraction from one surface to the other; (b) Shows an FGM that has more ceramic volume. fraction and less metal volume fraction; and (c) Shows an FGM that has equal volume fractions of both ceramic and metal.

The primary aim of the project is to formulate a finite element for FGM plates by using

HOST12. Many elements have been formulated for FGM plates in past. Here the aim is to formulate an element which is based on HOST12. Results will be compared with other theories such as FOST, TSDT and 3D elasticity solutions to find the difference in error. The objective of the work is to study the natural frequency characteristics of the pre-stressed functionally graded plates and to carry out parametric study.

In parametric study, the effects of thickness and aspect ratios on frequencies will be studied as well as influence of the initial membrane stresses on natural frequencies under various boundary conditions will be studied.

Vibration Analysis

The vibration problem can be formulated by using Hamilton's principle as follows:

$$\delta(U - T) = 0$$

where U is the Strain Energy and T is the Kinetic Energy.

In the *FE* analysis, U and T can be written as:

$$U = 1/2\{d\}^T [K_T] \{d\} \text{ and}$$

$$T = 1/2\ \omega^2 \{d\}^T [M] \{d\}$$

where $[K_T]$ and $[M]$ are the total structural stiffness matrix and mass Matrix respectively, ω is the natural frequency assuming harmonic time dependent displacement. The matrices $[K_T]$ and $[M]$ are obtained by Assembling the element stiffness and mass matrices respectively. The Governing equation for the natural vibration analysis (undamped) thus becomes

$$[K_T]\{d\} - \omega^2 [M]\{d\} = 0$$

Subspace Iteration Technique

The subspace iteration technique is designed to solve large eigenproblems. This technique has been developed for the solution of the p smallest eigenvalues and eigenvectors as mostly required by dynamic analysis. In present formulation, this technique is followed to solve eigen problem. The solution procedure was named subspace iteration technique, because the iteration is equivalent to iterating with a q-dimensional subspace and should not be regarded as simultaneous iteration with an individual vector.

In essence, the subspace iteration method developed by Bathe (Petyt, 1990) consists of the following three steps:

1. Establish q starting iteration vectors, $q = 2p$ where p is the number of required eigen values.
2. Use simultaneous inverse iteration on the q vectors and Ritz analysis to extract the best eigenvalue and eigenvector approximation from the q iteration vectors.
3. After iteration convergence, use the sturm sequence check to verify that the required eigen values and corresponding eigenvectors have been calculated.

The selection of starting iteration vector in step (1) and sturm sequence check in step (3) are considered important parts of the iteration procedures.

NUMERICAL INVESTIGATION AND RESULTS

Various numerical examples solved are described and discussed for establishing the accuracy of the various theories compared for

Table 1: Summary of Numerical Examples

S. No.	Type of Analysis	Description	Purpose
1.	Static analysis	Simply supported square plate	To test stiffness matrix
2.	Vibration analysis	Simply supported square plate without pre-stress	To test mass matrix and subspace iteration technique
3.	Vibration analysis	Simply supported square plate with uniform axial pre-stress	To test geometric stiffness matrix

free vibration analysis of FG plates. Table 1 describes the various case studies carried to evaluate the MATLAB program.

Case Study

Geometric Properties

Length, a: 4 m,

Width, b: 4 m and

Thickness, h: 1 m

Table 2 gives the various material properties used for the analysis of functionally graded plates.

This involves the following steps:

- Finite Element programming using MATLAB 7.2, since no commercial software package available supports the modeling of this special type of materials.
- Finding out the natural frequencies of FG plates and the influence of volume fraction index, initial stresses, aspect ratio and the slenderness ratio of the plate on the natural frequencies.

Table 2: The Exponential Variation of Young's Modulus

S. No.	Material	Young's Modulus, E(GPa)	Poisson's Ratio, ν	Density, ρ (kg/m^3)
1.	Aluminium	70	0.3	2707
2.	Ceramic (Zirconia)	151	0.3	3000
3.	FG Plate (Al + Ceramic)	70/151	0.3	2707/3000

Table 3: Comparison of Natural Frequencies from Various Theories

Mode	3D Elasticity Solution	HOST12	TSDT (Reddy and Cheng, 2001)	FOST (Mallikarjuna and Kant, 1988)
1	0.0932	0.0932 (0.0)	0.0930 (–0.21)	0.0929 (–0.25)
2	0.2226	0.2226 (0.0)	0.2220 (–0.27)	0.2218 (–0.33)
3	0.3421	0.3421 (0.0)	0.3406 (–0.44)	0.3403 (–0.52)
4	0.4171	0.4172 (0.0)	0.4151 (–0.48)	0.4145 (–0.63)
5	0.5239	0.5240 (0.0)	0.5208 (–0.59)	0.5202 (–0.71)
6	–	0.6573 (–)	0.6525 (–)	(–)
7	0.6889	0.6892 (0.0)	0.6839 (–0.73)	0.6827 (–0.90)

- Carrying out various case studies on the FE model.

Validation of the Results

The accuracy of the program has been verified by using the results available for isotropic and homogeneous aluminium plate. The results are validated based on the literature available. Table 3 gives this comparison of non-dimensional natural frequencies for first seven modes of isotropic aluminium plate obtained using HOST12 and other theories. It can be concluded that HOST12 results are almost equal to the 3D elasticity solutions. The error associated with TSDT is less as compared to FOST.

CONCLUSION

- The simple C0 iso-parametric finite element formulation of an assumed higher order displacement model HOST12, employed herein is more accurate than CPT, FOST and TSDT in predicting the natural frequencies of any functionally graded plates. The results obtained by HOST12 are almost equal to Three Dimensional elasticity solutions unlike other theories.
- In contrast to the FOST, the present HOST does not require a shear correction coefficient due to more realistic representation of the cross-sectional deformation
- On the basis of excellent agreement of the present results with analytical results for isotropic plates, it is fair to say that the present theory is accurate in predicting the natural frequencies of the functionally graded plates. The error in predicting lower modes is lower as compared to higher modes.
- The volume fraction index, initial stresses, aspect ratio and the slenderness ratio have significant influence on the natural frequencies of the functionally graded plates.
- As the volume fraction index increases, the natural frequencies of the functionally graded plates keep decreasing. The natural frequencies of the functionally graded plates lie in between the values of natural frequencies of the corresponding isotropic materials used in the manufacturing of the functionally graded plates.
- As expected, the natural frequencies of the Functionally graded plates with tensile in-plane load is predicted more than the plate without load and similarly, the natural frequencies of the Functionally graded plates with compressive in-plane load is predicted less than the plate without load.
- The second and third natural frequencies for the case of plate without load are almost same and similarly, the fifth and sixth frequencies are also same. Same is the situation for the plate with biaxial in-plane loading. However, this is not the situation for the case of plate with uni-axial in-plane load.
- As the thickness of FG plate decreases, the difference in the results by various theories decreases and hence the results remain the same irrespective of the theory used provided the plate is thin.

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